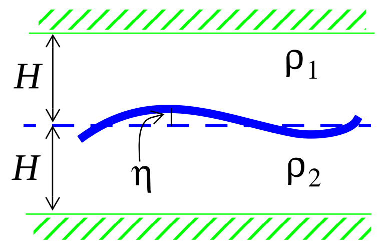
7. Baroclinic dynamics

7.1 Two-layer models

Consider a system consisting of two immiscible fluids of density ρ_1 and ρ_2, with $\rho_1 < \rho_2$. In the resting state, the fluids occupy layers of equal depth, H, as shown in the diagram. The system is bounded by rigid lids at the top and bottom.



Let η denote the displacement of the interface above the height H. Thus $H + \eta$ is the disturbed height of the interface. The flow is assumed in hydrostatic balance throughout. Therefore, in the upper layer,

$$p = p_s + \rho_1 g(2H - z) \ , \tag{7.1}$$

and in the lower layer,

$$p = p_s + \rho_1 g(H - \eta) + \rho_2 g(H + \eta - z) , \tag{7.2}$$

where p_s is the pressure at the upper boundary. It follows that the acceleration due to the pressure gradient force is $-(1/\rho_1)\nabla p_s$ in the upper layer and $-(1/\rho_2)\nabla p_s - [(\rho_2 - \rho_1)/\rho_2] g \nabla \eta$ in the lower layer. Both of these accelerations are independent of depth within the layer, so we can once again consistently assume that the horizontal flow is independent of depth within each layer.

We thus have shallow water dynamics in each of two layers:

$$\partial \boldsymbol{v}_1/\partial t + (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1 = -f\hat{\boldsymbol{z}} \times \boldsymbol{v}_1 - (1/\rho_1)\nabla p_s \tag{7.3}$$

$$\partial \boldsymbol{v}_2/\partial t + (\boldsymbol{v}_2 \cdot \nabla)\boldsymbol{v}_2 = -f\hat{\boldsymbol{z}} \times \boldsymbol{v}_2 - (1/\rho_2)\nabla p_s - g^*\nabla\eta , \tag{7.4}$$

where $g^* \equiv (\rho_2 - \rho_1)g/\rho_2$. The vorticity equations are:

$$\partial \zeta_1/\partial t + \boldsymbol{v}_1 \cdot \nabla\zeta_1 = -(f + \zeta_1)\nabla \cdot \boldsymbol{v}_1 - \beta v_1 \tag{7.5}$$

$$\partial \zeta_2/\partial t + \boldsymbol{v}_2 \cdot \nabla\zeta_2 = -(f + \zeta_2)\nabla \cdot \boldsymbol{v}_2 - \beta v_2 . \tag{7.6}$$

The coupling between the layers is due to the interface displacement η, which is related to the divergent velocity. Indeed, continuity of mass applied to the separate layers yields

$$\partial\eta/\partial t = -\nabla \cdot [(H+\eta)\boldsymbol{v}_2] = \nabla \cdot [(H-\eta)\boldsymbol{v}_1]\,. \tag{7.7}$$

Note that the divergence of the vertically integrated flow is zero, i.e.,

$$\nabla \cdot [(H+\eta)\boldsymbol{v}_2 + (H-\eta)\boldsymbol{v}_1] = 0\,, \tag{7.8}$$

as required by the rigid lids.

Geostrophy in this system implies $f\hat{\boldsymbol{z}} \times \boldsymbol{v}_1 = -(1/\rho_1)\nabla p_s$ and $f\hat{\boldsymbol{z}} \times \boldsymbol{v}_2 = -(1/\rho_2)\nabla p_s - g^*\nabla\eta$, so that

$$f\hat{\boldsymbol{z}} \times (\rho_1\boldsymbol{v}_1 - \rho_2\boldsymbol{v}_2) = (\rho_1 - \rho_2)g\nabla\eta\,. \tag{7.9}$$

That is, the pressure-gradient forces resulting from the slope of the interface must be balanced by the Coriolis force acting on a flow *with vertical shear.* The *y*-component of 7.9 is

$$\rho_1 u_1 - \rho_2 u_2 = -(\rho_2 - \rho_1)\frac{g}{f}\frac{\partial\eta}{\partial y}\,. \tag{7.10}$$

This balance is analogous to the thermal-wind equation in a continuously stratified model. Here it is essentially "Margules' relation" for the *discontinuity* in wind at a front.

In the Boussinesq approximation, we assume $\rho_2 - \rho_1 \ll \rho_2$, and Margules' relation becomes

$$f\hat{\boldsymbol{z}} \times (\boldsymbol{v}_1 - \boldsymbol{v}_2) = -\frac{g'}{f}\nabla\eta\,, \tag{7.11}$$

where we may define $g' \equiv g(\rho_2 - \rho_1)/\rho_0$ with $\rho_0 = (\rho_1 + \rho_2)/2$, say.

The potential vorticity equations for each layer follow easily from 7.5 - 7.7:

$$\frac{\partial}{\partial t}\left(\frac{f+\zeta_i}{h_i}\right) + \boldsymbol{v}_i \cdot \nabla\left(\frac{f+\zeta_i}{h_i}\right) = 0\,, \qquad i = 1, 2\,, \tag{7.12}$$

where $h_1 \equiv H - \eta$ and $h_2 \equiv H + \eta$.

7.2 Internal waves in the 2-layer model

In layer models, internal waves are interfacial waves. The linearized vorticity equations are

$$\frac{\partial\zeta_1}{\partial t} = -fD_1, \quad \frac{\partial\zeta_2}{\partial t} = -fD_2\,, \tag{7.13}$$

where $D_i \equiv \partial u_i/\partial x + \partial v_i/\partial y$. The linearized divergence equations are

$$\frac{\partial\eta}{\partial t} = -HD_2 = HD_1\,, \tag{7.14}$$

implying $D_1 = -D_2$. It then follows from 7.13 that $\partial(\zeta_1 + \zeta_2)/\partial t = 0$. Using the notation $\bar{A} = (A_1 + A_2)/2$ and $\hat{A} = (A_1 - A_2)/2$, we may write $\partial\bar{\zeta}/\partial t = 0$, $\partial\hat{\zeta}/\partial t = -f\hat{D}$, as well as

$$\frac{\partial \bar{D}}{\partial t} = f\bar{\zeta} - \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\nabla^2 p_s - \frac{\rho_1 - \rho_2}{2\rho_2} g \nabla^2 \eta \tag{7.15}$$

and

$$\frac{\partial \hat{D}}{\partial t} = f\hat{\zeta} - \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\nabla^2 p_s + \frac{\rho_1 - \rho_2}{2\rho_2} g \nabla^2 \eta . \tag{7.16}$$

A time derivative of 7.16 yields

$$\frac{\partial^2}{\partial t^2}\hat{D} = f\frac{\partial}{\partial t}\hat{\zeta} - \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\nabla^2 \frac{\partial p_s}{\partial t} + \frac{\rho_1 - \rho_2}{2\rho_2} g \nabla^2 \frac{\partial \eta}{\partial t} . \tag{7.17}$$

SInce $\bar{D} = 0$ implies $\partial^2 \bar{D}/\partial t^2 = 0$, and since $\partial\bar{\zeta}/\partial t = 0$, a time derivative of 7.15 yields

$$\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\nabla^2 \frac{\partial p_s}{\partial t} = -\frac{\rho_1 - \rho_2}{2\rho_2} g \nabla^2 \frac{\partial \eta}{\partial t} . \tag{7.18}$$

Making this substitution in 7.17 yields

$$\frac{\partial^2}{\partial t^2}\hat{D} = f\frac{\partial}{\partial t}\hat{\zeta} + \left[\frac{\rho_1 - \rho_2}{2\rho_2}\left(1 + \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)\right] g \nabla^2 \frac{\partial \eta}{\partial t} . \tag{7.19}$$

This can be simplified and written entirely in terms of $\hat{D}$ by using $\partial\eta/\partial t = -HD_2 = H\hat{D}$ and $\partial\hat{\zeta}/\partial t = -f\hat{D}$. Thus,

$$\frac{\partial^2}{\partial t^2}\hat{D} = -f^2\hat{D} + g^* H \nabla^2 \hat{D} . \tag{7.20}$$

This is exactly the same as the 1-layer shallow-water equation with g replaced by

$$g^* \equiv \frac{\rho_2 - \rho_1}{2\rho_0} g = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g , \tag{7.21}$$

the "reduced gravity". Some people like to think instead of an "equivalent depth" $H^* \equiv \frac{\rho_2 - \rho_1}{2\rho_0} H$.

For normal modes of the form $e^{i(kx + ly - \nu t)}$ on an *f*-plane, 7.20 provides the same dispersion relation as in the 1-layer case presented in section 6.1, namely $\nu^2 - f_0^{\,2} - g^* HK^2 = 0$. In addition to these gravity-inertia modes, steady geostrophic disturbances, with $\nu = 0$ and $\bar{D} = \hat{D} = 0$, are also solutions of the linearized system 7.13 - 7.14, just as in the 1-layer model. To realize the additional degree of freedom made possible by the second layer, one needs background vorticity gradients (*e.g.*, β), as will be seen in the next section.

In the ocean, we can associate level 1 with surface water, level 2 with bottom water and the interface with the thermocline. One has typically $(\rho_2 - \rho_1)/(2\rho_2) \sim 10^{-3}$ and so $H^* \sim$ a few meters. The radius of deformation $L_R = \sqrt{g^* H}/f_0 = \sqrt{gH^*}/f_0 \sim \mathrm{O}(50 \text{ km})$.

If we had utilized the Boussinesq approximation (an excellent approximation in the ocean) by setting $\rho_1 \approx \rho_2 \approx \rho_0$ in the surface pressure gradient terms in 7.15-7.16, we could derive 7.20 much more quickly, since the term $\nabla^2 p_s$ would be immediately eliminated from the equation for $\partial\hat{D}/\partial t$. The final result would be identical to 7.20.

If $(\rho_2 - \rho_1)/\rho_0 \ll 1$, one might be tempted to set $\rho_1 = \rho_2$ in the term $(\rho_1 - \rho_2) g \nabla^2 \eta$ as well. If one does this, one loses the internal gravity waves, of course.

Note that one needs the non-Boussinesq result $g^* \equiv g(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ in order to take the limit $\rho_1 \to 0$, in which the upper layer's effect on the lower layer must disappear, i.e., $g^* \to g$.

7.3 Quasi-geostrophy in the layer model

We can proceed either by applying the scaling arguments to the mass and momentum equations or by approximating the potential vorticity directly, as in the 1-layer case. We choose the second alternative.

In the upper layer,

$$\frac{f + \zeta_1}{H - \eta} \approx \frac{1}{H}\left(f + \zeta_1 + \frac{f_0 \eta}{H} \right) \tag{7.22}$$

if (1) $\zeta_1 \ll f$ (small Rossby number), (2) $\eta / H \ll 1$ (length scales not too much larger than $L_R \equiv \sqrt{g^* H} / f$) and (3) $f \approx f_0$ (channel of meridional extent much less than earth's radius). Similarly in the lower layer,

$$\frac{f + \zeta_2}{H + \eta} \approx \frac{1}{H}\left(f + \zeta_2 - \frac{f_0 \eta}{H} \right). \tag{7.23}$$

We now assume that the flow is geostrophic to lowest order, so that, from 7.9,

$$u_i = -\frac{\partial \psi_i}{\partial y}; \qquad v_i = \frac{\partial \psi_i}{\partial x}, \qquad i = 1, 2, \tag{7.24}$$

where the streamfunctions are given by $f_0 \psi_1 = p_s / \rho_1$ and $f_0 \psi_2 = p_s / \rho_2 + [(\rho_2 - \rho_1) / \rho_2] g \eta$, or

$$\frac{\rho_1 \psi_1 - \rho_2 \psi_2}{2 \rho_0} = -\frac{g^* \eta}{f_0}. \tag{7.25}$$

The Boussinesq approximation, which we adopt from here on, is

$$\frac{\psi_1 - \psi_2}{2} = \hat{\psi} = -\frac{g^* \eta}{f_0}. \tag{7.26}$$

The upper-layer potential vorticity becomes

$$q_1 = \frac{1}{H}\left[f + \nabla^2 \psi_1 - \frac{1}{2 L_R^2} (\psi_1 - \psi_2) \right], \tag{7.27}$$

where $L_R \equiv \frac{f_0^2}{g^* H}$, the "internal radius of deformation". In the lower layer, the potential vorticity is

$$q_2 = \frac{1}{H}\left[f + \nabla^2 \psi_1 + \frac{1}{2 L_R^2} (\psi_1 - \psi_2) \right]. \tag{7.28}$$

If we don't use the Boussinesq approximation, the only difference is that we would have $\frac{1}{2 L_R^2} \frac{\rho_1 \psi_1 - \rho_2 \psi_2}{\rho_0}$ for the third term in the brackets.

The conservation of potential vorticity becomes, to lower order,

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi_1 - \frac{1}{2 L_R^2} (\psi_1 - \psi_2) \right) + \boldsymbol{V}_g \cdot \nabla \left(\nabla^2 \psi_1 - \frac{1}{2 L_R^2} (\psi_1 - \psi_2) \right) = -\beta \frac{\partial \psi_1}{\partial x} \tag{7.29}$$

and

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_2 + \frac{1}{2L_R^2}(\psi_1 - \psi_2)\right) + \boldsymbol{V}_g \cdot \nabla\left(\nabla^2\psi_2 + \frac{1}{2L_R^2}(\psi_1 - \psi_2)\right) = -\beta\frac{\partial\psi_2}{\partial x}, \tag{7.30}$$

where $(\boldsymbol{V}_g)_i = \left(-\frac{\partial\psi_i}{\partial y}, \frac{\partial\psi_i}{\partial x}\right)$ and β is constant. More elegantly,

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \qquad i = 1, 2, \tag{7.31}$$

where $q_1 = \nabla^2\psi_1 - (2L_R^2)^{-1}(\psi_1 - \psi_2) + \beta y$ and $q_2 = \nabla^2\psi_2 + (2L_R^2)^{-1}(\psi_1 - \psi_2) + \beta y$.

If the advection terms are negligible, then 7.29 and 7.30 imply

$$\frac{\partial}{\partial t}\nabla^2\overline{\psi} = -\beta\frac{\partial\overline{\psi}}{\partial x}, \qquad \overline{\psi} \equiv \frac{1}{2}(\psi_1 + \psi_2) \tag{7.32}$$

and

$$\frac{\partial}{\partial t}\left(\nabla^2\hat{\psi} - \frac{1}{L_R^2}\hat{\psi}\right) = -\beta\frac{\partial\hat{\psi}}{\partial x}, \qquad \hat{\psi} \equiv \frac{1}{2}(\psi_1 - \psi_2). \tag{7.33}$$

On an infinite β-plane, 7.32 gives us the dispersion relation

$$\omega = -\frac{\beta k}{k^2 + l^2} \tag{7.34}$$

for the *barotropic* Rossby wave. In the barotropic wave, both layers are moving in unison, with no divergence (because of the rigid lid). The equation 7.33 for $\hat{\psi}$ yields

$$\omega = -\frac{\beta k}{k^2 + l^2 + 1/L_R^2} \tag{7.35}$$

for the frequency of *baroclinic*, or internal, Rossby waves. In these solutions, the flows are exactly out of phase in the two layers and the interface is oscillating.

7.4 Baroclinic instability in the 2-layer model

If the zonal flows in the two layers are dependent only on y [$u_1 = u_1(y)$ and $u_2 = u_2(y)$], if the interface slope is such that this flow is geostrophically balanced,

$$\frac{u_1 - u_2}{2} = \frac{g^*}{f}\frac{\partial\eta}{\partial y} \tag{7.36}$$

and if $v_1 = v_2 = 0$, then the flow is an exact solution of the quasi-geostrophic equations of motion 7.31 (as well as the original shallow-water equations, for that matter).

But is this flow stable? That is, if we perturb it, will the perturbation grow? And if there are growing perturbations, what is their structure?

We denote the above basic state with tildes, i.e., $u_i = \tilde{u}_i(y)$, $v_i = \tilde{v}_i \equiv 0$. The basic-state potential vorticity in the upper layer, for example, is

$$\tilde{q}_1 = f + \nabla^2\tilde{\psi}_1 - \frac{1}{2L_R^2}(\tilde{\psi}_1 - \tilde{\psi}_2). \tag{7.37}$$

The gradient of PV plays an important role in the theory. In the upper layer it is

$$\frac{d\tilde{q}_1}{dy} = \beta - \frac{d^2}{dy^2}\tilde{u}_1 + \frac{1}{2L_R^2}(\tilde{u}_1 - \tilde{u}_2), \tag{7.38}$$

while in the lower layer,

$$\frac{d\tilde{q}_2}{dy} = \beta - \frac{d^2}{dy^2}\tilde{u}_2 - \frac{1}{2L_R^2}(\tilde{u}_1 - \tilde{u}_2). \tag{7.39}$$

We use primes to denote the deviation from the basic state. For example, $\psi_i = \tilde{\psi}_i(y) + \psi_i'(x, y, t)$. The equations governing *small-amplitude* deviations are

$$\frac{\partial}{\partial t}q_i' = -\tilde{u}_i\frac{\partial}{\partial x}q_i' - v_i'\frac{d\tilde{q}_i}{dy}, \qquad i = 1, 2, \tag{7.40}$$

where the potential vorticity perturbations are $q_1' = \nabla^2\psi_1' - (\psi_1' - \psi_2')/(2L_R^2)$ and $q_2' = \nabla^2\psi_2' + (\psi_1' - \psi_2')/(2L_R^2)$. When 7.40 is written entirely in terms of the perturbation streamfunction, we get, for the upper layer,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla^2\psi_1' - \frac{1}{2L_R^2}(\psi_1' - \psi_2')\right) = {}& -\tilde{u}_1\frac{\partial}{\partial x}\left(\nabla^2\psi_1' - \frac{1}{2L_R^2}(\psi_1' - \psi_2')\right) \\ & -\frac{\partial\psi_1'}{\partial x}\left(\beta - \frac{d^2\tilde{u}_1}{dy^2} + \frac{1}{2L_R^2}(\tilde{u}_1 - \tilde{u}_2)\right),\end{aligned} \tag{7.41}$$

and, for the lower layer,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla^2\psi_2' + \frac{1}{2L_R^2}(\psi_1' - \psi_2')\right) = {}& -\tilde{u}_2\frac{\partial}{\partial x}\left(\nabla^2\psi_2' + \frac{1}{2L_R^2}(\psi_1' - \psi_2')\right) \\ & -\frac{\partial\psi_2'}{\partial x}\left(\beta - \frac{d^2\tilde{u}_2}{dy^2} - \frac{1}{2L_R^2}(\tilde{u}_1 - \tilde{u}_2)\right).\end{aligned} \tag{7.42}$$

These are coupled linear equations for the evolution of ψ_1' and ψ_2'.

Since the coefficients are independent of *x*, we can write

$$\psi_i' = \mathrm{Re}\sum_{j=0}^{\infty}\psi_i^{(j)}(y, t)e^{ik_jx}, \qquad i = 1, 2, \tag{7.43}$$

where $k_j = 2\pi j/L$ and L is the length of the latitude circle. Substituting into 7.41 - 7.42, we obtain a pair of coupled equations for each *j*.

The $j = 0$, or zonally symmetric, part of the perturbation satisfies a particularly simple equation in quasi-geostrophic theory, since $v^{(j=0)} = (\partial/\partial x)\psi^{(j=0)} = 0$. Therefore, $\partial q_i^{(0)}/\partial t = -\tilde{u}_i\partial q_i^{(0)}/\partial x - v_i^{(0)}d\tilde{q}_i/dy = 0$. By perturbing the zonally symmetric part of the flow, one is simply allowing a slightly different zonal flow that is itself an exact solution of the nonlinear equations. Viewed as a perturbation, this time-independent part of the flow is obviously stable. We can therefore restrict attention to zonally asymmetric perturbations ($j > 0$).

If we assume channel geometry, we must have $v' = \partial\psi'/\partial x = 0$ at $y = 0$ and $y = L_y$, the latitudes of the channel walls. This must be satisfied separately by each zonal wave component if it is to hold for all x. But since $v^{(j)} = ik_j\psi^{(j)}$, the boundary condition is

$$\psi^{(j)} = 0 \quad \text{at} \quad y = 0, L_y \tag{7.44}$$

for all $j \neq 0$, that is, for all j's of interest.

Consider now this simplest special case (Phillips' model):

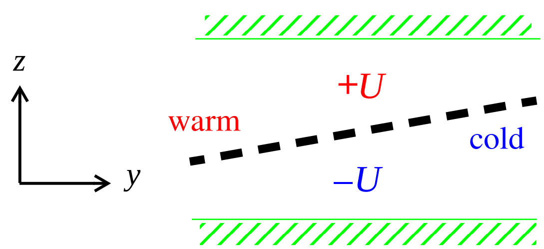
$$\tilde{u}_1 = U, \qquad \tilde{u}_2 = -U, \tag{7.45}$$

with U a constant. (This case differs from *any* other with y-independent basic flow by a Galilean transformation.) We have

$$\frac{d\tilde{q}_1}{dy} = \beta + \frac{U}{L_R^2}, \qquad \frac{d\tilde{q}_2}{dy} = \beta - \frac{U}{L_R^2} \tag{7.46}$$

and $d\tilde{\eta}/dy = (f/g^*)U$.

The case of particular interest is $U > 0$. For $f > 0$, the situation is shown in the diagram below. The westerly wind increases with height, as is typical in middle latitudes.



Since the coefficients in 7.40 are independent of y, we can separate variables and, using the boundary condition 7.44, write

$$\psi_i^{(j)}(y,t) = \sum_{m=1}^{\infty} \psi_i^{(j,m)}(t)\sin(l_m y), \tag{7.47}$$

where $l_m = m\pi/L_y$. Substituting a particular mode (j, m) into 7.40 produces

$$\frac{\partial\tilde{q}_i^{(j,m)}}{\partial t} + ikU\tilde{q}_i^{(j,m)} = -ik\tilde{\psi}_i^{(j,m)}\left(\beta \pm \frac{\dot{U}}{L_R^2}\right), \tag{7.48}$$

where (+) in the parentheses refers to the upper layer and (-) refers to the lower layer. With $K^2 \equiv k_j^2 + l_m^2$, the expressions for the $\tilde{q}_i$ are $\tilde{q}_1^{(j,m)} = -(K^2 + 0.5/L_R^2)\tilde{\psi}_1^{(j,m)} + (0.5/L_R^2)\tilde{\psi}_2^{(j,m)}$ and $\tilde{q}_2^{(j,m)} = -(K^2 + 0.5/L_R^2)\tilde{\psi}_2^{(j,m)} + (0.5/L_R^2)\tilde{\psi}_1^{(j,m)}$.

Since the coefficients in 7.40 are also independent of time, we can look for solutions of the form

$$\tilde{\psi}_i^{(j,m)}(t) = \underset{\sim}{\psi}_i^{(j,m)} e^{-i\omega t} = \underset{\sim}{\psi}_i^{(j,m)} e^{-ikct}, \tag{7.49}$$

where ω and $c = \omega/k$ are in general complex: $\omega = \omega_R + i\omega_I$ (ω_R and ω_I real). We generally have no guarantee that there will be solutions of this form, but if there are, and if there is at least one with $\omega_I > 0$, then we have discovered an exponentially growing instability; for the contribution of this mode to the streamfunction is

$$\psi_i = \sin(ly)\mathrm{Re}[\underset{\sim}{\psi}_i^{(j,m)} e^{i(kx - \omega_R t)}] e^{\omega_I t}, \tag{7.50}$$

which represents a wave propagating with phase speed c_R and simultaneously growing exponentially in amplitude, with the e-folding time $1/\omega_I = 1/(kc_I)$.

Substituting 7.49 into 7.48 and dropping the superscripts (j, m) for convenience, we have

$$(U - c)\left[-\left(K^2 + \frac{1}{2L_R^2}\right)\underset{\sim}{\psi}_1 + \frac{1}{2L_R^2}\underset{\sim}{\psi}_2\right] = -\left(\beta + \frac{U}{L_R^2}\right)\underset{\sim}{\psi}_1 \tag{7.51}$$

and

$$(-U - c)\left[-\left(K^2 + \frac{1}{2L_R^2}\right)\underset{\sim}{\psi}_2 + \frac{1}{2L_R^2}\underset{\sim}{\psi}_1\right] = -\left(\beta - \frac{U}{L_R^2}\right)\underset{\sim}{\psi}_2, \tag{7.52}$$

or, in terms of $\underset{\sim}{\bar{\psi}} = (1/2)(\underset{\sim}{\psi}_1 + \underset{\sim}{\psi}_2)$ and $\underset{\sim}{\hat{\psi}} = (1/2)(\underset{\sim}{\psi}_1 - \underset{\sim}{\psi}_2)$,

$$(U - c)\left[-K^2\underset{\sim}{\bar{\psi}} - \left(K^2 + \frac{1}{L_R^2}\right)\underset{\sim}{\hat{\psi}}\right] = -\left(\beta + \frac{U}{L_R^2}\right)(\underset{\sim}{\bar{\psi}} + \underset{\sim}{\hat{\psi}}) \tag{7.53}$$

and

$$(-U - c)\left[-K^2\underset{\sim}{\bar{\psi}} + \left(K^2 + \frac{1}{L_R^2}\right)\underset{\sim}{\hat{\psi}}\right] = -\left(\beta - \frac{U}{L_R^2}\right)(\underset{\sim}{\bar{\psi}} - \underset{\sim}{\hat{\psi}}). \tag{7.54}$$

Forming (7.53 + 7.54)/2, we get

$$\underset{\sim}{\bar{\psi}}(\beta + cK^2) = \underset{\sim}{\hat{\psi}}UK^2 \tag{7.55}$$

while (7.53 - 7.54)/2 yields

$$\underset{\sim}{\hat{\psi}}\left[\beta + c\left(K^2 + \frac{1}{L_R^2}\right)\right] = \underset{\sim}{\bar{\psi}}U\left(K^2 - \frac{1}{L_R^2}\right). \tag{7.56}$$

If $U = 0$ we have the two Rossby waves discussed earlier, with $\underset{\sim}{\hat{\psi}}$ corresponding to 7.34 (with $\underset{\sim}{\bar{\psi}} = 0$) and $\underset{\sim}{\bar{\psi}}$ corresponding to 7.35 (with $\underset{\sim}{\hat{\psi}} = 0$).

First consider the special case $\beta = 0$ (and $U \neq 0$). Then $c\underset{\sim}{\bar{\psi}} = U\underset{\sim}{\hat{\psi}}$ and $c(K^2 + 1/L_R^2)\underset{\sim}{\hat{\psi}} = U(K^2 - 1/L_R^2)\underset{\sim}{\bar{\psi}}$, or

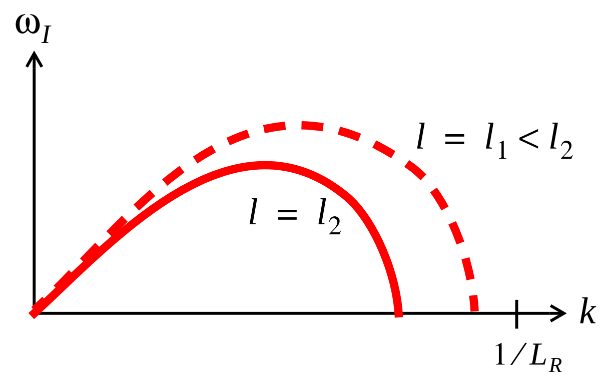
$$c^2 = U^2\left(\frac{K^2 - 1/L_R^2}{K^2 + 1/L_R^2}\right). \tag{7.57}$$

Therefore, if $K > 1/L_R$, both solutions for c are real: $c = \pm U\sqrt{(K^2 - 1/L_R^2)/(K^2 + 1/L_R^2)}$. But if $K < 1/L_R$, then c is pure imaginary and the two solutions yield one exponentially growing and one exponentially decaying wave.

The growing wave is the one of particular interest. We see that the flow under consideration is, in fact, unstable for any U (this is for $\beta = 0$, remember), and is unstable to perturbations with wavelength larger than $2\pi L_R$. The growth rate is proportional to U:

$$\omega_I = kc_I = kU\sqrt{\frac{k^2 + l^2 - 1/L_R^2}{k^2 + l^2 + 1/L_R^2}}. \tag{7.58}$$

It is easy to show that for a given k, increasing l always decreases ω_I, as illustrated below. There-



fore, the most unstable wave will have the smallest possible l (hence the largest possible meridional scale) and a value of k somewhat less than L_R^{-1}. Unstable waves with $\beta = 0$ have $c_R = 0$, which means that they move with the average of the two basic-state velocities as they grow.

For any two wave amplitudes $\underset{\sim}{A}$ and $\underset{\sim}{B}$, the x-t structure is $A' = \mathrm{Re}(\underset{\sim}{A}e^{ik(x-ct)})$ and $B' = \mathrm{Re}(\underset{\sim}{B}e^{ik(x-ct)})$. In general, $\underset{\sim}{A} = C\underset{\sim}{B}$, where $C = \alpha e^{i\varphi}$ and α and φ are real. Then the wave A' lags the wave B' by the angle φ:

$$A' = \mathrm{Re}(C\underset{\sim}{B}e^{ik(x-ct)}) = \alpha\mathrm{Re}(\underset{\sim}{B}e^{ik(x+L\varphi/(2\pi)-ct)}), \tag{7.59}$$

where $L = 2\pi/k$, the wavelength.

For the unstable wave above, $ic_I\bar{\psi} = U\hat{\psi}$ or $\hat{\psi} = i(c_I/U)\bar{\psi} = e^{i\pi/2}(c_I/U)\bar{\psi}$, so that $\hat{\psi}'$ lags $\bar{\psi}'$ by $\pi/2$ radians if $c_I > 0$. Since $\underset{\sim}{\bar{v}} = ik\underset{\sim}{\bar{\psi}}$, $\hat{\psi}'$ and $\bar{v}'$ are exactly in phase in the growing wave. For the decaying wave, $\hat{\psi}'$ leads $\bar{\psi}'$ by $\pi/2$ radians, and $\hat{\psi}'$ and $\bar{v}'$ ar exactly out of phase.

For the neutral waves ($K > 1/L_R$), $\hat{\psi}'$ and $\bar{\psi}'$ are in phase, and $\hat{\psi}'$ and $\bar{v}'$ are $\pi/2$ out of phase.

Since $\hat{\psi} = -(g^*/f)\eta'$, the interface displacement η' must be exactly out of phase with the mean meridional velocity perturbation $\bar{v}'$ in growing waves. This means that the wave is transporting lighter fluid poleward -- since poleward moving columns of fluid have a relatively low interface and are, therefore, relatively light.

In the ocean it is often a good approximation to set $\rho = \rho^* - \alpha T$, where T is the temperature, ρ^* is the density at $0°\mathrm{C}$ and α is a constant, the coefficient of thermal expansion. One is often interesting in the vertically integrated meridional flux of heat averaged over x (or zonal wavelength):

$$\mathcal{H} = c_p\int dz \int dx\ (v'T'). \tag{7.60}$$

Denoting x-averaging by { }, we have

$$\mathcal{H} = \frac{c_p}{\alpha}\{(H-\eta')v_1'(\rho^*-\rho_1)+(H+\eta')v_2'(\rho^*-\rho_1)\}$$
$$= \frac{c_p}{\alpha}\{-\eta' v_1'(\rho^*-\rho_1)+\eta' v_2'(\rho^*-\rho_1)\}. \tag{7.61}$$

But $\eta' v_1' = f(2g^*)^{-1}(\psi_2'-\psi_1')\partial\psi_1'/\partial x$. Therefore,

$$(2g^*/f)\{\eta' v_1'\} = \left\{\psi_2'\frac{\partial\psi_1'}{\partial x}\right\} = -\left\{\psi_1'\frac{\partial\psi_2'}{\partial x}\right\} = \left\{(\psi_2'-\psi_1')\frac{\partial\psi_2'}{\partial x}\right\}, \tag{7.62}$$

so that $\{\eta' v_1'\} = \{\eta' v_2'\} = \{\eta' \bar{v}'\}$ and

$$\mathcal{H} = -\frac{c_p}{\alpha}(\rho_2-\rho_1)\{\eta'\bar{v}'\}. \tag{7.63}$$

As expected, if η' is $180°$ out phase with $\bar{v}'$, then the wave is transporting heat poleward ($\mathcal{H} > 0$). To summarize all cases,

- growing waves transport heat poleward
- decaying waves transport heat equatorward
- neutral waves do not transport heat

As we saw above,

$$\{\eta'\bar{v}'\} = -\frac{f}{2g^*}\left\{\psi_1'\frac{\partial\psi_2'}{\partial x}\right\}. \tag{7.64}$$

Therefore, if a disturbance tilts to the west with increasing height, so that ψ_1 lags ψ_2 by an angle between 0 and π, then the angle between ψ_1 and $\partial\psi_2'/\partial x$ will be less than $\pi/2$ in absolute value, and the heat flux will be poleward. If a disturbance tilts eastward with height, it transports heat equatorward.

Returning again to the Phillips instability problem, now consider the case $\beta \neq 0$. Eqs. 7.55 - 7.56 imply

$$(\beta + cK^2)\left[\beta + c\left(K^2+\frac{1}{L_R^2}\right)\right] = U^2K^2\left(K^2-\frac{1}{L_R^2}\right). \tag{7.65}$$

Defining $c_1 = -\beta/K^2$ and $c_2 = -\beta/(K^2+1/L_R^2)$, we have $(c-c_1)(c-c_2) = U^2(K^2-1/L_R^2)(K^2+1/L_R^2)$ or, from the quadratic formula,

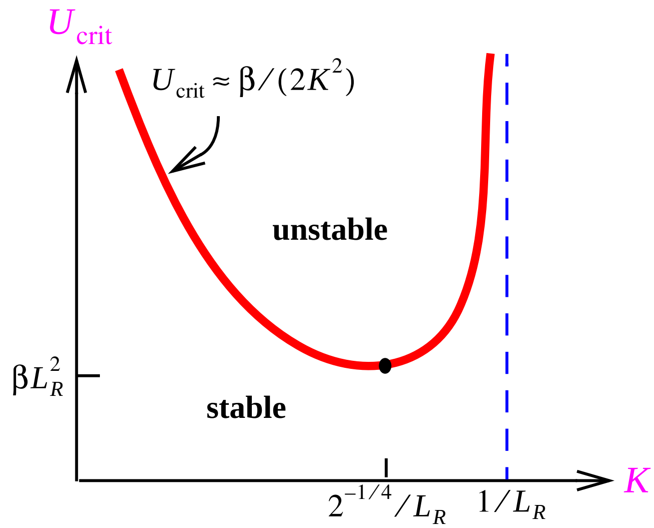
$$c = \frac{1}{2}(c_1+c_2) \pm \frac{1}{2}\left[(c_1+c_2)^2 - 4\left(c_1c_2 - U^2\frac{K^2-1/L_R^2}{K^2+1/L_R^2}\right)\right]^{1/2}. \tag{7.66}$$

We get instability if the factor inside the square root sign is negative, *i.e.*, if $(c_1-c_2)^2 < 4U^2(L_R^{-2}-K^2)/(L_R^{-2}+K^2)$. Once again, we see immediately from 7.66 that there are no unstable waves for $K^2 > 1/L_R^2$, for then the rhs of the inequality would be negative and the inequality could not hold.

Upon substituting the definitions of c_1 and c_2, and manipulating a bit, the instability criterion becomes

$$U > U_{\text{crit}} \equiv \frac{1}{2}\beta\frac{L_R^2}{(L_R K)^2}\frac{1}{\sqrt{1-(L_R K)^4}} \tag{7.67}$$

(where we have assumed that $U > 0$). As a function of *K*, the minimum value of U_{crit} is βL_R^2, occurring when $(KL_R)^4 = 1/2$.



Thus, two new features appear for $\beta \neq 0$:

- The flow is stable to all wavenumbers if $U < \beta L_R^2$.
- For any *U*, the flow is stable to short disturbances as well as very long ones.

For *U* only slightly greater than βL_R^2, the instability has a well-defined total wavenumber of approximately $1/(2^{1/4}L_R)$.

In midlatitudes, $\beta \approx f/a \approx (10^{-4}\ \text{sec}^{-1})/(7\times10^6\ \text{m}) \approx 1.5\times10^{-11}\ \text{m}^{-1}\text{sec}^{-1}$. Therefore, if $L_R \approx 10^6\ \text{m}$ (a typical choice for atmospheric applications, though it is difficult to make an intelligent choice without considering a model with greater vertical resolution), then $U_{\text{min}} \equiv \beta L_R^2 \approx 15\ \text{m/sec}$.

U_{min} has an interesting latitudinal dependence:

$$U_{\text{min}} = \frac{2\Omega\cos\theta_0}{a}\frac{g^*H}{(2\Omega\sin\theta_0)^2} \propto \frac{\cos\theta_0}{(\sin\theta_0)^2}. \tag{7.68}$$

Therefore, U_{min} increases rapidly as θ_0 decreases and unrealistically large vertical shears would be required in order to get this kind of instability in the tropics.

Note that for $0 < U < \beta L_R^2$, the potential vorticity *gradients* in the upper layer,

$$\frac{d\tilde{q}_1}{dy} = \beta + \frac{U}{L_R^2}, \tag{7.69}$$

and lower layer,

$$\frac{d\tilde{q}_2}{dy} = \beta - \frac{U}{L_R^2}, \tag{7.70}$$

are both positive, being dominated by β. The appearance of unstable solutions when $U > \beta L_R^2$ coincides with $d\tilde{q}_2/dy$ becoming negative.

Stability when $0 < U < \beta L_R^2$ can be understood as a special case of a theorem for general $\tilde{u}_1(y)$ and $\tilde{u}_2(y)$, the so-called Charney-Stern necessary condition for normal-mode instability. It is analogous to the Rayleigh-Kuo theorem for barotropic shear instability presented in section 6.10. The proof given below is more physically grounded than the one in chapter 6.

Note first that in an unstable normal mode, in which all variables have the form $\varphi' = \mathrm{Re}\{\underset{\sim}{\varphi}(y)e^{ik(x-ct)}\} = \frac{1}{2}[\underset{\sim}{\varphi}(y)e^{ik(x-ct)} + \underset{\sim}{\varphi}^*(y)e^{-ik(x-c^*t)}]$, with $c_I \equiv \mathrm{Im}\{c\} > 0$, we have

$$\{q'^2\} = \frac{1}{2}\left|\underset{\sim}{q}(y)\right|^2 e^{2kc_I t}, \tag{7.71}$$

so that the zonally averaged eddy potential enstrophy increases everywhere simultaneously.

The Charney-Stern theorem states that this kind of growth of a perturbation is impossible if $d\tilde{q}_1/dy$ and $d\tilde{q}_2/dy$ are of the same sign everywhere. From 7.40, we have

$$\frac{1}{2}\frac{\partial}{\partial t}q_i'^2 = -\tilde{u}_1\frac{1}{2}\frac{\partial}{\partial x}q_i'^2 - v_i'q_i'\frac{d\tilde{q}_i}{dy}, \tag{7.72}$$

so that

$$\frac{1}{2}\frac{\partial}{\partial t}\{q_i'^2\} = -\{v_i'q_i'\}\frac{d\tilde{q}_i}{dy}, \quad i = 1, 2. \tag{7.73}$$

If an unstable normal mode exists, then in light of 7.71, 7.73 implies that the flux $\{v_i'q_i'\}$ must be directed down the local potential-vorticity gradient everywhere, i.e., $\{v_i'q_i'\}$ must be everywhere of opposite sign to $d\tilde{q}_i/dy$.

We assume, as before, a channel geometry, with $v' = 0$ at $y = 0, L$. Since $q_1' = \nabla^2\psi_1' - (2L_R^2)^{-1}(\psi_1' - \psi_2')$ and $v_1' = \partial\psi_1'/\partial x$, we get, for the flux in the upper layer,

$$\{v_1'q_1'\} = \frac{\partial}{\partial y}\left\{\frac{\partial\psi_1'}{\partial x}\frac{\partial\psi_1'}{\partial y}\right\} + \frac{1}{2L_R^2}\left\{\frac{\partial\psi_1'}{\partial x}\psi_2'\right\} \tag{7.74}$$

(using the periodicity in *x* several times), and, in the bottom layer,

$$\{v_2'q_2'\} = \frac{\partial}{\partial y}\left\{\frac{\partial\psi_2'}{\partial x}\frac{\partial\psi_2'}{\partial y}\right\} + \frac{1}{2L_R^2}\left\{\frac{\partial\psi_2'}{\partial x}\psi_1'\right\}. \tag{7.75}$$

Adding these, and using the periodicity one more time, we reach

$$\{v_1'q_1'\} + \{v_2'q_2'\} = -\frac{\partial}{\partial y}\{u_1'v_1' + u_2'v_2'\}, \tag{7.76}$$

because $u_1' = -\partial\psi_1'/\partial y$. Finally, we integrate in *y* to obtain

$$\int_0^L (\{v_1'q_1'\} + \{v_2'q_2'\}) = -\{u_1'v_1' + u_2'v_2'\}\Big|_0^L = 0. \tag{7.77}$$

The last equality follows from the boundary conditions at the channel walls. We conclude that $\{v'q'\}$ cannot be of the same sign everywhere -- otherwise it could not integrate to zero. Since this flux must be downgradient everywhere in an unstable mode, $d\tilde{q}/dy$ cannot be of the same sign everywhere, either. That is, a sign change in the PV gradient is necessary for instability. This proves the Charney-Stern theorem for normal modes.

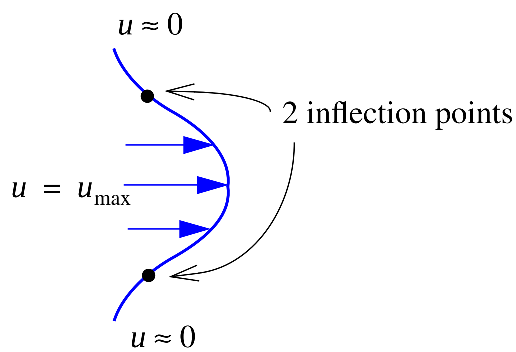
In general, from 7.39,

$$\frac{d\tilde{q}_1}{dy} = \beta - \frac{d^2\tilde{u}_1}{dy^2} + \frac{1}{2L_R^2}(\tilde{u}_1 - \tilde{u}_2) \tag{7.78}$$

and

$$\frac{d\tilde{q}_2}{dy} = \beta - \frac{d^2\tilde{u}_2}{dy^2} - \frac{1}{2L_R^2}(\tilde{u}_1 - \tilde{u}_2). \tag{7.79}$$

If we have just one homogeneous layer, or if $\tilde{u}_1 = \tilde{u}_2$, and if the flow is not rotating ($\beta = 0$), the theorem reduces to the statement that $d^2\tilde{u}/dy^2$ must change sign for there to be a possibility of a normal mode instability, as previously noted in section 6.10. Since a jet-like profile will have two such *inflection points*, such a profile is always a candidate for instability.



In the rotating case, however, the quantity that must change sign is $\beta - d^2\tilde{u}/dy^2$. Since β is positive, it tends to stabilize a barotropic westerly jet. A jet-like profile can be stable if its vorticity gradients are small compared with β, the vorticity gradient of the solid-body rotation.

When the potential vorticity gradient changes sign primarily because of the term $d^2\tilde{u}/dy^2$, the resulting instability is often termed a "barotropic instability", while if the change in sign is due, rather, to the vertical shear, $\tilde{u}_1 - \tilde{u}_2$ (as in our previous problem), the instability is termed a "baroclinic instability".

Sometimes the distinction between these two types of instability is made from energetic considerations -- see the next section. In any case, if $\tilde{u}$ is independent of *y*, the perturbation problem is referred to as "purely baroclinic"; if $\tilde{u}$ is independent of *z*, it is referred to as "purely barotropic". In general, if $\tilde{u} = \tilde{u}(y, z)$, the problem is "mixed barotropic-baroclinic".

7.5 Energetics and diagnostics of the 2-layer QG model

Before examining the energetics of these unstable waves, it is useful to back up and consider the 2-layer quasi-geostrophic vorticity and height equations from which the potential vorticity equation that we have been using actually derives.

Without going through the detailed scaling analysis again, we can see by exact analogy with the 1-layer model that the vorticity equations are, to first order in Rossby number,

$$\frac{\partial \zeta_1}{\partial t} = -J(\psi_1, \zeta_1) - \beta \frac{\partial \psi_1}{\partial x} - f_0 D_1 \tag{7.80}$$

in the upper layer and

$$\frac{\partial \zeta_2}{\partial t} = -J(\psi_2, \zeta_2) - \beta \frac{\partial \psi_2}{\partial x} - f_0 D_2 \tag{7.81}$$

in the lower layer, where $\zeta_i = \nabla^2 \psi_i$, the lowest (0th) order vorticity and $D_i = \nabla \cdot \boldsymbol{V}_i$, the lowest (1st) order divergence. (Warning: subscripts here refer to the model level, not to the order in the Rossby number expansion.)

But 7.8, to lowest-order in Rossby number, reduces to $D_1 = \nabla \cdot \boldsymbol{V}_1 = -D_2 = -\nabla \cdot \boldsymbol{V}_2$, since η / H is O(Ro) if $L / L_R \le O(1)$, as discussed in chapter 5. Setting $D = D_2 = -D_1$ for convenience, we have

$$\frac{\partial \zeta_1}{\partial t} = -J(\psi_1, \zeta_1) - \beta \frac{\partial \psi_1}{\partial x} + f_0 D \tag{7.82}$$

and

$$\frac{\partial \zeta_2}{\partial t} = -J(\psi_2, \zeta_2) - \beta \frac{\partial \psi_2}{\partial x} - f_0 D, \tag{7.83}$$

and in particular,

$$\frac{\partial}{\partial t}\hat{\zeta} = -\tfrac{1}{2}[J(\psi_1, \zeta_1) - J(\psi_2, \zeta_2)] - \beta \frac{\partial}{\partial x}\hat{\psi} + f_0 D. \tag{7.84}$$

The continuity equation in the lower layer, to first order in Rossby number, is

$$\frac{\partial \eta}{\partial t} = -J(\psi_2, \eta) - HD, \tag{7.85}$$

precisely as in the 1-layer case. Recall that $\eta = -\frac{f_0}{g^*}\hat{\psi}$. Similarly, in the upper layer, $\partial(H - \eta)/\partial t = -J(\psi_1, H - \eta) - HD_1$, or

$$\frac{\partial \eta}{\partial t} = -J(\psi_1, \eta) - HD, \tag{7.86}$$

which is identical to 7.85 except for the streamfunction.

But $J(\psi_1, \eta) = (f_0/g^*)J(\psi_1, \psi_2 - \psi_1) = (f_0/g^*)J(\psi_1, \psi_2)$ while, for the lower layer, $J(\psi_2, \eta) = (f_0/g^*)J(\psi_2, \psi_2 - \psi_1) = -(f_0/g^*)J(\psi_2, \psi_1)$. From the antisymmetry of J, we see that 7.85 and 7.86 are, in fact, consistent.

As before, the potential vorticity equations are obtained by eliminating D from the vorticity and continuity equations. Forming (7.82)–(f_0/H) (7.85) yields the PV equation in the upper layer, and (7.83)–(f_0/H) (7.86) in the lower layer.

The continuity equation is more commonly written

$$\frac{\partial \eta}{\partial t} = -J(\overline{\psi}_1, \eta) - HD, \tag{7.87}$$

which is equivalent to 7.85 and 7.86. Writing η in terms of $\hat{\psi}$ and operating with ∇^2, we get

$$\frac{\partial}{\partial t}\nabla^2\hat{\psi} = -\nabla^2 J(\overline{\psi}, \hat{\psi}) + \frac{g^* H}{f_0}\nabla^2 D. \tag{7.88}$$

It follows from 7.84 and 7.88 that

$$\left(\nabla^2 - \frac{1}{L_R^2}\right)D = \frac{f_0}{g^* H}\left[\nabla^2 J(\overline{\psi}, \hat{\psi}) - \frac{1}{2}(J(\psi_1, \nabla^2\psi_1) + J(\psi_2, \nabla^2\psi_2)) - \beta\frac{\partial}{\partial x}\hat{\psi}\right], \tag{7.89}$$

which is the *omega equation* for the 2-layer model.

Let us assume a channel geometry and denote an integration over x and y by square brackets. Thus, $[A] = \int_0^{L_y}\int_0^{L_x} A\,dx\,dy$. Then 7.87 implies

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\eta^2] = -H[D\eta]. \tag{7.90}$$

By forming $[\psi_1 \cdot (7.82)]$ and $[\psi_2 \cdot (7.83)]$, we also have

$$\frac{\partial}{\partial t}[\tfrac{1}{2}|\nabla\psi_1|^2] = -f_0[\psi_1 D] \tag{7.91}$$

and

$$\frac{\partial}{\partial t}[\tfrac{1}{2}|\nabla\psi_2|^2] = f_0[\psi_2 D], \tag{7.92}$$

respectively, or

$$\frac{\partial}{\partial t}[\tfrac{1}{2}(\tfrac{1}{2}|\nabla\psi_1|^2 + \tfrac{1}{2}|\nabla\psi_2|^2)] = -f_0[\hat{\psi}D] = g^*[\eta D]. \tag{7.93}$$

Let us define $\mathrm{KE} = (\rho_0/2)(|\nabla\psi_1|^2/2 + |\nabla\psi_2|^2/2)$ to be the vertically averaged kinetic energy per unit volume [if we were not working in the Boussinesq system, we would have instead $\mathrm{KE} = (\rho_1/4)|\nabla\psi_1|^2 + (\rho_2/4)|\nabla\psi_2|^2$]. Let us also define the averaged available potential energy per unit volume as $\mathrm{APE} = (\rho_0 g^*/(2H))[\eta^2] = (\rho_0/(2L_R^2))[\hat{\psi}^2]$. Then

$$\frac{\partial}{\partial t}\mathrm{KE} = \rho_0 g^*[\eta D] \tag{7.94}$$

and

$$\frac{\partial}{\partial t}\mathrm{APE} = -\rho_0 g^*[\eta D], \tag{7.95}$$

so that $\frac{\partial}{\partial t}(\mathrm{KE{+}APE}) = 0$.

In analogy with the 1-layer case, the available potential energy is the total potential energy minus the potential energy of the state with $\eta = 0$. Since the latter part is constant, we should be able to show that 7.95 also applies to the total potential energy. We have

$$\begin{aligned}\frac{1}{2H}\int_0^{2H} g\rho z\,dz &= \frac{g}{2H}\left(\rho_2\int_0^{H+\eta} z\,dz + \rho_1\int_{H+\eta}^{2H} z\,dz\right)\\ &= \frac{g}{2H}\left[\frac{\rho_2}{2}(H^2 - 2H\eta + \eta^2) + \frac{\rho_1}{2}(4H^2 - H^2 + 2H\eta - \eta^2)\right] \qquad (7.96)\\ &= \alpha_1 + \alpha_2\eta + \frac{1}{2H}\left[\frac{g}{2}(\rho_2 - \rho_1)\eta^2\right],\end{aligned}$$

where α_1 and α_2 are constants. Since $\frac{\partial}{\partial t}[\eta] = 0$ from conservation of mass, 7.96 implies

$$\frac{\partial}{\partial t}\mathrm{PE} = \frac{\rho_0 g^*}{2H}\frac{\partial}{\partial t}[\eta^2] = \frac{\partial}{\partial t}\mathrm{APE}, \tag{7.97}$$

as expected.

From 7.94-7.95, we see that the conversion from potential energy to kinetic energy is $\rho_0 g^*[\eta D]$; that is, kinetic energy is generated when relatively light (warm) fluid ($\eta < 0$) is rising ($D < 0$ implies convergence in the lower layer).

To compute $[\eta' D']$ for our unstable waves, we first need D'. We consider only the special case $\beta = 0$. One can obtain D' from 7.89, but this is rather laborious. Since we have already solved for the phase speed $c(K)$, it is easier to use the perturbation form of the continuity equation 7.87:

$$\frac{\partial}{\partial t}\eta' = -\tilde{\bar{u}}\frac{\partial}{\partial x}\eta' - \bar{v}'\frac{\partial}{\partial y}\tilde{\eta} - HD'. \tag{7.98}$$

Now $\tilde{\bar{u}} = \frac{1}{2}(\tilde{u}_1 + \tilde{u}_2) = \frac{1}{2}(U - U) = 0$, so

$$H\underset{\sim}{D} = ik\left(c\underset{\sim}{\eta} - \frac{\partial\tilde{\eta}}{\partial y}\underset{\sim}{\bar{\psi}}\right) = ik\left(-\frac{cf_0}{g^*}\underset{\sim}{\hat{\psi}} - \frac{f_0}{g^*}U\underset{\sim}{\bar{\psi}}\right), \tag{7.99}$$

or $\frac{g^* H}{f_0}\underset{\sim}{D} = -ik(c\underset{\sim}{\hat{\psi}} + U\underset{\sim}{\bar{\psi}})$. But from 7.55, $c\underset{\sim}{\bar{\psi}} = U\underset{\sim}{\hat{\psi}}$, so

$$\frac{g^* H}{f_0}\underset{\sim}{D} = -ikc\underset{\sim}{\hat{\psi}}\left(1 + \frac{U^2}{c^2}\right) \tag{7.100}$$

or

$$H\underset{\sim}{D} = ikc\underset{\sim}{\hat{\eta}}(1 + U^2/c^2). \tag{7.101}$$

For neutral waves ($K > 1/L_R$), c is real and, therefore, D' and ψ' (or D' and η') are $\pi/2$ out of phase, so $[D'\eta'] = 0$. For the growing waves, $c = ic_I,\ c_I > 0,$ we have

$$H\underset{\sim}{D} = -kc_I(1 - U^2/c_I^2)\underset{\sim}{\eta} = \frac{2k/L_R^2}{K^2 + 1/L_R^2}\sqrt{\frac{1/L_R^2 - K^2}{K^2 + 1/L_R^2}}. \tag{7.102}$$

Therefore D' and η' are exactly in phase, $[D'\eta'] > 0$, and the wave is converting potential into kinetic energy.

Finally, let us consider the energetics of the linear waves. Total eddy energy is not necessarily conserved because unstable waves tap into the energy of the basic state. For an arbitrary basic zonal flow, $\tilde{u}_1(y)$ and $\tilde{u}_2(y)$, the linearized continuity equation is

$$\frac{\partial\eta'}{\partial t} = -\tilde{\bar{u}}\frac{\partial\eta'}{\partial x} - \bar{v}\frac{\partial\tilde{\eta}}{\partial y} - HD', \tag{7.103}$$

while the linearized vorticity equations (from 7.82-7.83) are

$$\frac{\partial\zeta_1'}{\partial t} = -\tilde{\bar{u}}\frac{\partial\zeta_1'}{\partial x} - \bar{v}\frac{\partial\tilde{\zeta}_1}{\partial y} - \beta v_1' + f_0 D' \tag{7.104}$$

$$\frac{\partial \zeta_1'}{\partial t} = -\tilde{u}\frac{\partial \zeta_1'}{\partial x} - \bar{v}\frac{\partial \tilde{\zeta}_1}{\partial y} - \beta v_1' + f_0 D' . \tag{7.105}$$

Multiplying 7.103 by η' and averaging over the channel domain yields

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\eta'^2) = -\left[\eta' v' \frac{\partial \tilde{\eta}}{\partial y}\right] - H[\eta' D'] , \tag{7.106}$$

while multiplying 7.104 and 7.105 by ψ_1' and ψ_2', respectively, and adding gives

$$\frac{\partial}{\partial t}[\tfrac{1}{2}(\tfrac{1}{2}|\nabla\psi_1'|^2 + \tfrac{1}{2}|\nabla\psi_2'|^2)] = \tfrac{1}{2}\left[\tilde{u}_1\psi_1'\frac{\partial \zeta_1'}{\partial x} + \tilde{u}_2\psi_2'\frac{\partial \zeta_2'}{\partial x}\right] - f_0[\hat{\psi}D'] . \tag{7.107}$$

But $[\tilde{u}_i\psi_i'(\partial\zeta_i'/\partial x)] = -[\tilde{u}_i(\partial\psi_i'/\partial x)\zeta_i'] = -[\tilde{u}_i v_i'\zeta_i']$, and we have already seen, in deriving 7.74 - 7.75, that $\{v_i'\zeta_i'\} = -\partial\{u_i'v_i'\}/\partial y$, where $\{\}$ denotes integration over *x*. Therefore,

$$[\tilde{u}_i\psi_i'(\partial\zeta_i'/\partial x)] = +\left[\tilde{u}_i\frac{\partial}{\partial y}(u_i'v_i')\right] = -\left[\frac{\partial \tilde{u}_i}{\partial y}(u_i'v_i')\right] . \tag{7.108}$$

If we define the eddy kinetic energy as

$$\mathrm{EKE} = \frac{\rho_0}{2}[\tfrac{1}{2}|\nabla\psi_1'|^2 + \tfrac{1}{2}|\nabla\psi_2'|^2] \tag{7.109}$$

and the eddy available potential energy as

$$\mathrm{EAPE} = \frac{\rho_0 g^*}{2H}[\eta'^2] = \frac{\rho_0}{2L_R^2}[\hat{\psi}'^2] , \tag{7.110}$$

then 7.107 is

$$\frac{\partial}{\partial t}\mathrm{EKE} = -\rho_0\left[\sum_{i=1,2}\frac{\partial \tilde{u}_i}{\partial y}(u_i'v_i')\right] + \rho_0 g^*[\eta' D'] \tag{7.111}$$

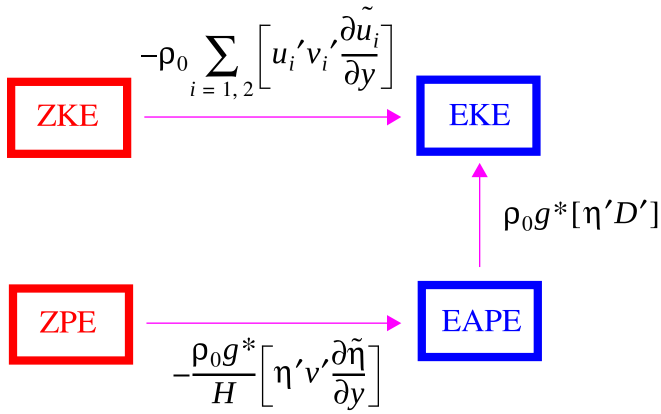
and 7.106 is

$$\frac{\partial}{\partial t}\mathrm{EAPE} = -\frac{\rho_0 g^*}{H}\left[\eta'\bar{v}'\frac{\partial \tilde{\eta}}{\partial y}\right] - \rho_0 g^*[\eta' D'] . \tag{7.112}$$

From 7.111 - 7.112,

$$\frac{\partial}{\partial t}(\mathrm{EKE} + \mathrm{EAPE}) = -\rho_0\left[\sum_{i=1,2}\frac{\partial \tilde{u}_i}{\partial y}(u_i'v_i')\right] - \frac{\rho_0 g^*}{H}\left[\eta'\bar{v}'\frac{\partial \tilde{\eta}}{\partial y}\right] . \tag{7.113}$$

The first term on the rhs represents barotropic production of eddy energy. The second term represents baroclinic production.



Barotropic production occurs when the flux of zonal momentum, $u'v'$, is down the mean velocity gradient on the average. Baroclinic production occurs when $\eta'v'$ is down the mean height gradient (*i.e.*, when the heat flux is down the mean temperature gradient) on average.

Note that in a purely baroclinic problem, in which $\tilde{u}$ is not a function of *y*, there is no barotropic production. Therefore, in an unstable wave, the conversion $[\eta'D']$ must be positive, and the baroclinic production must be positive and larger than the conversion, in order that EKE and EAPE both increase.

Problems

7.1

7.2

7.3

7.4