The Depth-Dependent Current and Wave Interaction Equations: A Revision

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ABSTRACT

This is a revision of a previous paper dealing with three-dimensional wave-current interactions. It is shown that the continuity and momentum equations in the absence of surface waves can include waves after the addition of three-dimensional radiation stress terms, a fairly simple alteration for numerical ocean circulation models. The velocity that varies on time and space scales, which are large compared to inverse wave frequency and wavenumber, is denoted by $\hat{u}_\alpha$ and, by convention, is called the "current." The Stokes drift is labeled $u_{S\alpha}$ and the mean velocity is $U_\alpha \equiv \hat{u}_\alpha + u_{S\alpha}$. When vertically integrated, the results here are in agreement with past literature.

Surface wind stress is empirical, but transfer of the stress into the water column is a function derived in this paper. The wave energy equation is derived, and terms such as the advective wave velocity are weighted vertical integrals of the mean velocity. The wave action equation is not an appropriate substitute for the wave energy equation when the mean velocity is depth dependent.

1. Introduction

Mellor (2003, hereafter M03) produced an analysis providing depth-dependent wave-current interaction equations which, when vertically integrated, were in agreement with the depth-integrated equations of Longuet-Higgins and Stewart (1962, 1964, hereafter L-HS), Phillips (1977), and others. Nevertheless, a commentary by Ardhuin et al. (2008a) pointed to a discrepancy in M03 for shallow water ($kD \cong 1$) compared with a case of unforced waves traversing a bottom with variable topography. This led to a further discovery that, with the M03 formulation, unforced waves with bottom variations produced mean currents even for deep water (say, $kD \cong 10$), a physically unacceptable finding.

The present paper, although containing elements of M03, abandons the a priori use of sigma coordinates; characterization of waves derived for a flat bottom can be misinterpreted in the sigma domain. Specific differences between M03 and the present paper are postponed to the summary in section 7.

A recent paper by Smith (2006), starting from the vertically integrated equations of motion, explores the interaction between wave momentum and current momentum; this has significant instructional value. McWilliams et al. (2004) and Ardhuin et al. (2008b) develop equations for the current $\hat{u}_\alpha$; their analyses are complicated, and it is hard to see correspondence to the results of the present paper. Here, we obtain depth-dependent equations corresponding to the vertically integrated equations of L-HS and Phillips (1977). There is emphasis on developing equations that are easily incorporated into three-dimensional circulation models. It is shown that these models as now coded require only the addition of depth-dependent stress radiation terms to the momentum equation. Of course, a wave model is required to supply wave energy and wavenumber. The wave model can also provide variables for a wave-sensitive surface wind stress parameterization (Donelan 1990). A finding in M03 and here is that transport of the surface stress into the water column is supported by pressure and turbulence, not turbulence alone as, for example, in Mellor and Yamada (1982), Large et al. (1994), and many other papers.

Section 2 contains the derivation of the continuity and momentum equations that includes waves. Use is made of an elemental control volume bounded by material surfaces vertically and surfaces normal to the Cartesian coordinates horizontally. Current plus wave velocities—set equal to the standard linear solutions—are used to evaluate the continuity and momentum bal-

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ances for the elemental control volume and the results are phase averaged. Special care is required to evaluate the balance of pressure forces following closely the reasoning of L-HS. Section 3 deals with the vertical transport of surface wind stress. The wave energy equation is presented in section 4. The transformation of the Cartesian equations to a sigma coordinate version is in section 5. In section 6, the aforementioned wave-current interaction case of Ardhuin et al. (2008a) is discussed; unlike M03, there is agreement with their results and the results of this paper.

2. Derivation in Cartesian coordinates

To construct equations that include wave motions, the waves are conventionally represented by the linear irrotational solutions for elevation $\tilde{\eta}$, velocity $(\tilde{u}_\alpha, \tilde{w})$, and kinematic pressure $\tilde{p}$ (dynamic pressure divided by a reference water density, ρ_0), as follows:

$$\tilde{\eta} = a\cos\psi, \tag{1a}$$

$$(\tilde{u}_\alpha, \tilde{w}) = kac\left(\frac{k_\alpha}{k}\frac{\cosh k(z+h)}{\sinh kD}\cos\psi, \frac{\sinh k(z+h)}{\sinh kD}\sin\psi\right), \tag{1b), (1c}$$

$$\tilde{p} = kac^2\frac{\cosh k(z+h)}{\sinh kD}\cos\psi = ga\frac{\cosh k(z+h)}{\cosh kD}\cos\psi. \tag{1d}$$

The Cartesian coordinates are (x_β, z), where Greek subscripts α or β denote horizontal coordinates; the vertical coordinate z is positive upward from the sea surface. In (1), $\psi = k_\beta x_\beta - \omega t$; k_β and ω are directional wavenumber and frequency, such that $\omega = \sigma + k_\beta \hat{U}_\beta$ and $k = |k_\beta|$; σ is the intrinsic frequency and $\hat{U}_\alpha$ is the Doppler velocity; a is wave amplitude; $c = \sigma/k$ is the phase speed; h is the bottom depth; and $\hat{\eta}$ is the mean surface elevation. The wave elevation is defined by (1a) and $\eta \equiv \hat{\eta} + \tilde{\eta}$. The water column mean depth is $D \equiv \hat{\eta} + h$. The change from the first to the second form in (1d) uses the dispersion relation $\sigma^2 = gk\tanh kD$ and is an example of similar manipulations below. Note that $\tilde{p}(\hat{\eta}) = g\tilde{\eta}$.

The vertical locations of material surfaces are

$$s(x, y, z, t) = z + \tilde{s}, \quad \tilde{s} = a\frac{\sinh k(z+h)}{\sinh kD}\cos\psi. \tag{2a), (2b}$$

Equation (2b) is obtained from $\partial\tilde{s}/\partial t = \tilde{w}$. The vertical derivatives are

$$s_z \equiv \frac{\partial s}{\partial z} = 1 + \tilde{s}_z, \quad \tilde{s}_z = ak\frac{\cosh k(z+h)}{\sinh kD}\cos\psi, \tag{3a), (3b}$$

and the horizontal derivatives are

$$s_\alpha \equiv \frac{\partial s}{\partial x_\alpha} = \tilde{s}_\alpha = -ak\frac{\sinh k(z+h)}{\sinh kD}\sin\psi. \tag{4a), (4b}$$

Note that, at $z = \hat{\eta}$, (2b) yields $\tilde{s} = a\cos\psi = \tilde{\eta}$, whereas at $z = -h$, $\tilde{s} = \tilde{s}_\alpha = 0$.

In the derivation of the above equations, ka, $\partial h/\partial x_\alpha$, $\partial a/\partial x_\alpha$, and $\partial k_\beta/\partial x_\alpha$ are assumed to be small. In the following nonlinear analyses, the same quantities are also assumed to be small (properly nondimensionalized on a representative k and σ). In particular, we note that terms additional to (1) and therefore (2), (3), and (4) that account for bottom slope are proportional to $ka(\partial h/\partial x_\alpha)$. To obtain this scaling, start with the linear irrotational wave equations; then expand the potential function using the small parameter, $\varepsilon = \partial h/\partial x$. The lowest-order solutions are (1a)–(1c) and the next order that satisfies a nonzero but small bottom slope yields the aforementioned scaling. Further analysis, or indeed intuition, reveals that a more specific parameter is $ka(\partial h/\partial x)/\sinh kh$ because for deep water, the bottom slope should not be a factor in the description of surface gravity waves. Toward the final nonlinear equations derived below, terms of order $(ka)^4$ are neglected relative to retained terms of order $(ka)^2$. For variable topography, it is assumed that $(ka)^2[(\partial h/\partial x)/\sinh kh]^2$ is small; this could be a problem for small kh; however, see section 6.

The integral control volume equation for mass conservation (density is constant) is

$$\oint \dot{u}_k n_k \, dA = 0, \tag{5}$$

where $n_k = (n_\alpha, n_z)$ is the unit vector normal to elemental surfaces dA of the control volume. The velocity $\dot{u}_k$ is relative to moving boundaries of the control volume.

The horizontal components of the integral momentum equation is

$$\frac{\partial}{\partial t}\int u_\alpha \, dV + \oint u_\alpha \dot{u}_k n_k \, dA - \int \varepsilon_{\alpha\beta z} f_z u_\beta \, dV = -\int \frac{\partial p}{\partial x_\alpha} dV + \oint \tau_\alpha \, dA. \tag{6}$$

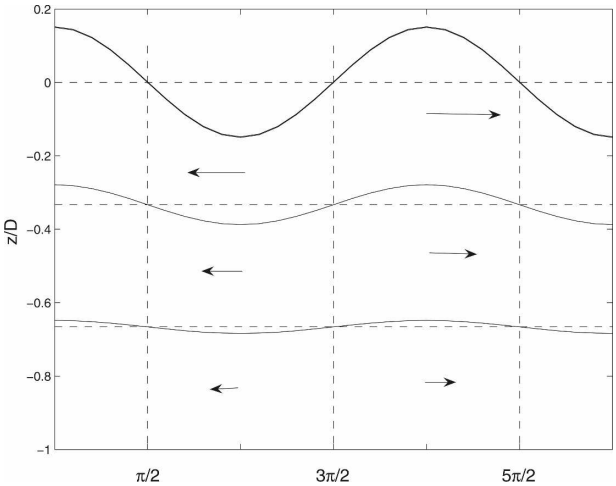

FIG. 1. A flow schematic. The solid lines are material surfaces. Due to increased velocity magnitude and increased flow area, the volume flow magnitude below crests exceeds that below troughs, resulting in Stokes drift.

The elemental volume is dV, and $\oint \tau_\alpha \, dA$ represents momentum transfer by pressure and turbulence as described in section 3. (The form $\int \partial p/\partial x_\alpha \, dV$ will be more convenient than the equivalent $\oint p n_\alpha \, dA$.)

a. The velocity terms

Velocity components are divided such that

$$u_\alpha(z, s, t) = \hat{u}_\alpha(z) + \tilde{u}_{e\alpha}(s, t),$$
$$w(z, s, t) = \hat{w}(z) + \tilde{w}_e(s, t), \qquad \text{(7a), (7b)}$$

where $(\hat{u}_\alpha, \hat{w})$ are defined to be "current" velocities that vary on spatial and temporal scales that are large compared to k^{-1} and σ^{-1}. The wave velocities are extrapolated from z to $z + \tilde{s}$ such that

$$\tilde{u}_{e\alpha}(s, t) \equiv \tilde{u}_\alpha(z, t) + \left.\frac{\partial \tilde{u}_\alpha}{\partial z}\right|_z \tilde{s}, \quad \tilde{w}_e(s, t) \equiv \tilde{w} + \left.\frac{\partial \tilde{w}}{\partial z}\right|_z \tilde{s}. \qquad \text{(8a), (8b)}$$

Equations (7) together with (8) are not novel; it is accepted that the absolute velocity on the crest of a wave exceeds the absolute velocity in the trough—a fact that is intrinsic to Stokes drift. The wave velocities [Eqs. (8a) and (8b)] and material surfaces [Eqs. (2a) and (2b)] are schematically shown in Fig. 1.

Overbars will represent phase averaging: $\overline{(\ )} = (2\pi)^{-1}\int_0^{2\pi}(\ )d\psi$ (so that, e.g., $\overline{\cos\psi} = \overline{\sin\psi} = 0$, $\overline{\cos^2\psi} = \overline{\sin^2\psi} = 1/2$, $\overline{\cos\psi \sin\psi} = 0$, etc.).

Consider a control volume bounded by the surfaces, x, $x + \Delta x$; y, $y + \Delta y$ and s, $s + s_z\Delta z$. On the s surfaces, the unit vectors are $(n_\beta, n_z) = (s_\beta, \sqrt{1 - s_\beta^2}) \cong (s_\beta, 1 - s_\beta^2/2)$. Applying (5) to this control volume and phase averaging, one obtains

$$\frac{\partial}{\partial x_\alpha}[\overline{u_\alpha s_z}] + \frac{\partial}{\partial z}[\overline{(w + u_\beta \tilde{s}_\beta)s_z}] = 0, \qquad (9)$$

after $\Delta x \Delta y \Delta z$ has been factored out of the equation. Notice that $(\partial f/\partial s)\Delta s = (\partial f/\partial z)(\partial z/\partial s)s_z\Delta z = (\partial f/\partial z)\Delta z$.

Now $\overline{u_\alpha s_z} = \overline{(\hat{u}_\alpha + \tilde{u}_{e\alpha})s_z} = \hat{u}_\alpha + \overline{\tilde{u}_{e\alpha}s_z}$, so that

$$\overline{u_\alpha s_z} = \hat{u}_\alpha + u_{S\alpha}, \qquad (10a)$$

where

$$u_{S\alpha} = \overline{\tilde{u}_{e\alpha}s_z} = \overline{\left(\tilde{u}_\alpha + \frac{\partial \tilde{u}_\alpha}{\partial z}\tilde{s}\right)(1 + \tilde{s}_z)} = \frac{\partial \overline{\tilde{u}_\alpha \tilde{s}}}{\partial z} \qquad (10b)$$

is the Stokes drift. Thus, the Stokes drift is the product of the phase-averaged horizontal component of the extrapolated velocity, $\tilde{u}_\alpha + (\partial \tilde{u}_\alpha/\partial z)_z\tilde{s}$, and the flow area, $(1 + \tilde{s}_z)\Delta z \Delta y$ (if $\alpha = x$). This is illustrated in Fig. 1. Further, using (1b) and (2b),

$$u_{S\alpha} = \frac{\partial \overline{\tilde{u}_\alpha \tilde{s}}}{\partial z} = \frac{(ka)^2 c}{2}\frac{k_\alpha}{k}\frac{\cosh 2k(z + h)}{\sinh^2 kD}$$
$$= \frac{2k_\alpha E}{c}\frac{\cosh 2k(z + h)}{\sinh 2kD}. \qquad (10c)$$

The latter form uses the dispersion relation $\sigma^2 = kg \tanh kD$ and the definition of wave energy $E = g\overline{\tilde{\eta}^2} = ga^2/2$ [see (A.8)]. For deep water, $u_{S\alpha} = (2k_\alpha E/c) \exp[2k(z - \hat{\eta})]$.

For the second term in (9),

$$\overline{(w + u_\beta \tilde{s}_\beta)s_z} = \overline{(\hat{w} + \tilde{w}_e)s_z} = \hat{w},$$

because, using (7a), (8a), (3), and (4), $\overline{u_\beta \tilde{s}_\beta s_z} = 0$. Now define

$$U_\alpha \equiv \hat{u}_\alpha + u_{S\alpha}, \quad W \equiv \hat{w} \qquad \text{(11a), (11b)}$$

(the last substitution for cosmetic uniformity). Therefore, the continuity Eq. (9), is simply

$$\frac{\partial U_\alpha}{\partial x_\alpha} + \frac{\partial W}{\partial z} = 0. \qquad (12)$$

The momentum Eq. (6) applied to the same control volume as above and for the horizontal advective and Coriolis portions of (6) are

$$\text{ADV} = \frac{\partial}{\partial t}[\overline{u_\alpha s_z}] + \frac{\partial}{\partial x_\beta}[\overline{u_\beta u_\alpha s_z}] + \frac{\partial}{\partial z}[\overline{(w + u_\beta s_\beta)u_\alpha s_z}]$$
$$- \varepsilon_{\alpha\beta z} f_z \overline{u_\beta s_z}, \qquad (13)$$

after phase averaging and factoring out $\Delta x \Delta y \Delta z$.

The velocity in the first (tendency) term and last (Coriolis) term of (13) is U_α as determined in (10a) and (11a). Now consider the second bracketed term,

$$\overline{u_\beta u_\alpha s_z} = \overline{(\hat{u}_\beta + \tilde{u}_{e\beta})(\hat{u}_\alpha + \tilde{u}_{e\alpha})s_z}$$
$$= \hat{u}_\beta \hat{u}_\alpha + \hat{u}_\beta \overline{\tilde{u}_{e\alpha} s_z} + \hat{u}_\alpha \overline{\tilde{u}_{e\beta} s_z} + \overline{\tilde{u}_\beta \tilde{u}_\alpha},$$

or

$$\overline{u_\beta u_\alpha s_z} = U_\beta U_\alpha + \overline{\tilde{u}_\beta \tilde{u}_\beta}. \quad (14)$$

In arriving at (14), a term of order $(ka)^4$ relative to terms of order $(ka)^2$ has been neglected. For the third bracketed term in (13),

$$\overline{(\hat{w} + \tilde{w}_e)u_\alpha s_z + u_\beta \tilde{s}_\beta u_\alpha s_z} = \hat{w}\overline{s_z u_\alpha} = WU_\alpha. \quad (15)$$

Using (14) and (15), (13) may be written as

$$\mathrm{ADV} = \frac{\partial}{\partial t}(U_\alpha) + \frac{\partial}{\partial x_\beta}(U_\beta U_\alpha + \overline{\tilde{u}_\beta \tilde{u}_\beta}) + \frac{\partial}{\partial z}(WU) - \varepsilon_{\alpha\beta z} f_z U_\beta. \quad (16)$$

b. The pressure terms

Dealing with the pressure term in (6) is complicated. From the vertical component of momentum, we have

$$\frac{\partial p}{\partial z} + \frac{\partial \tilde{w}^2}{\partial z} + \frac{\partial \tilde{w}}{\partial t} + g = 0.$$

For the region $-h < z < \hat{\eta}$, the mean hydrostatic equation is

$$\frac{\partial \hat{p}}{\partial z} + \frac{\partial \overline{\tilde{w}^2}}{\partial z} + g = 0. \quad (17)$$

After integration, (17) yields

$$\hat{p} + \overline{\tilde{w}^2} + g(z - \hat{\eta}) = p_{\mathrm{atm}}; \quad -h < z < \hat{\eta}. \quad (18)$$

Equations (17) and (18) were derived by L-HS and Phillips (1977). The kinematic atmospheric pressure (divided by the reference water density) is p_{atm}. [As a check, $\hat{p}(\hat{\eta}) + \overline{\tilde{w}^2(\hat{\eta})} = p_{\mathrm{atm}}$ is obtained by applying the integral momentum equation to a thin control volume that includes the air–sea interface.]

In M03, the wave pressure field was treated similarly to the velocity as in (7) and (8) and as illustrated in Fig. 1, where the velocity field fills the entire region, $\eta > z > -h$, but a similar interpretation is not possible for wave pressure given by (1a) and (1d); see Fig. 2. Note that $\tilde{p}(\hat{\eta}) - g\tilde{\eta} = 0$, so that wave pressure is nil at the surface. In the shaded region of Fig. 2, denoted by $-|\tilde{\eta}| < z - \hat{\eta} < |\tilde{\eta}|$, the pressure is evidently hydrostatic, as noted by L-HS, and the entire field can be described by

$$p = \hat{p} + \tilde{p}_e, \quad \tilde{p}_e = \begin{cases} g(\eta - z), & -|\tilde{\eta}| < z - \hat{\eta} < |\tilde{\eta}| \\ \tilde{p}(z), & -h < z < \hat{\eta}, \end{cases} \quad (19a), (19b)$$

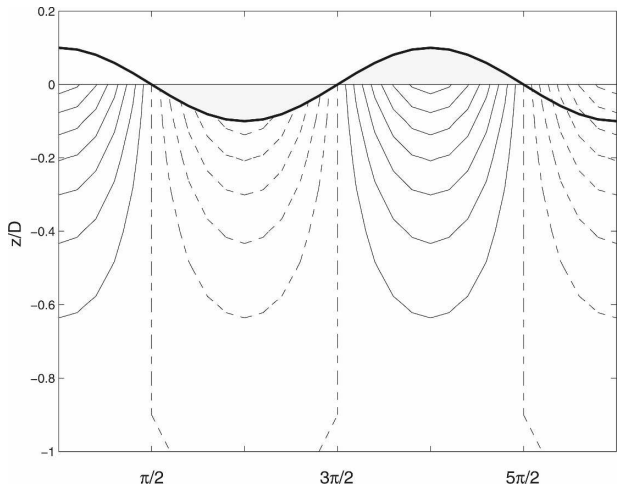

FIG. 2. The pressure field. Below $z = 0$ (here $\hat{\eta}$ is set to zero), solid lines are contours of constant pressure (solid lines are positive $\tilde{p}$, dashed lines are negative) according to (1d), which, at $z = 0$, supports hydrostatic pressure in the shaded regions; i.e., $\tilde{p}(z = 0) = g\tilde{\eta}$.

where $\hat{p}(z)$ is given by (18) and $\tilde{p}(z)$ by (1d). In a trough, overlapping regions as given by (19) are conceptually unattractive but are nevertheless dictated by (1a) and (1d).

Because $\hat{p}$ and $\tilde{p}$ are functions of z (and not s), the phase-averaged contribution of $\overline{\partial \tilde{p}(z)/\partial x_\alpha}$ for $-h < z < \hat{\eta}$ is nil. However, in the region $-|\tilde{\eta}| < z - \hat{\eta} < |\tilde{\eta}|$, a phase-averaged integral of (19) exists and is

$$\overline{\int_{\hat{\eta}}^{\hat{\eta}+\tilde{\eta}} g(\eta - z)\, dz} = g\frac{\overline{\tilde{\eta}^2}}{2} = \frac{E}{2} \quad \text{at} \quad z = \hat{\eta}, \quad (20)$$

so that using (18),

$$\overline{\frac{\partial p}{\partial x_\alpha}} = \frac{\partial \hat{p}}{\partial x_\alpha} + \frac{\partial E_D}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}(g\eta - \overline{\tilde{w}^2} + p_{\mathrm{atm}} + E_D), \quad (21)$$

where a modified Dirac delta function is defined such that

$$E_D = 0 \quad \text{if} \quad z \neq \hat{\eta} \quad \text{and} \quad \int_{-h}^{\hat{\eta}+} E_D\, dz = E/2. \quad (22)$$

[In a finite difference rendering of E_D, the top vertical layer of incremental size, δz—and only the top layer—would be occupied by $\partial E_D/\partial x_\beta = (\delta z)^{-1}\partial(E/2)/\partial x_\beta$.]

c. The phase-averaged momentum equation

Inserting (16) and (21) into (6)—after factoring out $\Delta x \Delta y \Delta z$—yields

$$\frac{\partial}{\partial t}(U_\alpha) + \frac{\partial}{\partial x_\beta}(U_\alpha U_\beta) + \frac{\partial}{\partial z}(WU_\alpha) - \varepsilon_{\alpha\beta z} f_z U_\beta = -\int^{\hat{\eta}} \frac{\partial b}{\partial x_\alpha} dz - \frac{\partial}{\partial x_\alpha}(g\hat{\eta} + p_{\mathrm{atm}}) - \frac{\partial}{\partial x_\beta} S_{\alpha\beta} + \frac{\partial \overline{\tau}_\alpha}{\partial z}, \tag{23}$$

where

$$S_{\alpha\beta} = \overline{\tilde{u}_\alpha \tilde{u}_\beta} + \delta_{\alpha\beta}(-\overline{\tilde{w}^2} + E_D), \tag{24a}$$

which is implicit in the L-HS derivation after vertical integration.

As in M03, it is convenient to define the following terms:

$$F_{\mathrm{SS}} \equiv \frac{\sinh k(z+h)}{\sinh kD}, \quad F_{\mathrm{CS}} \equiv \frac{\cosh k(z+h)}{\sinh kD},$$

$$F_{\mathrm{SC}} \equiv \frac{\sinh k(z+h)}{\cosh kD}, \quad F_{\mathrm{CC}} \equiv \frac{\cosh k(z+h)}{\cosh kD}.$$

{For deep water ($kD \gg 1$), $F_{\mathrm{SS}} = F_{\mathrm{CS}} = F_{\mathrm{SC}} = F_{\mathrm{CC}} = \exp[k(z - \hat{\eta})]$.} Then, substituting (1) into (24a) yields

$$S_{\alpha\beta} = kE\left(\frac{k_\alpha k_\beta}{k^2} F_{\mathrm{CS}} F_{\mathrm{CC}} - \delta_{\alpha\beta} F_{\mathrm{SC}} F_{\mathrm{SS}}\right) + \delta_{\alpha\beta} E_D. \tag{24b}$$

In (23), the buoyancy term, where $b \equiv g\hat{\rho}/\rho_0$, has been added; it could have been included in (13), but was omitted to simplify the subsequent discussion. It is assumed that the waves are not affected by buoyancy, or more precisely, that $N^2/\sigma^2 \ll 1$ in regions occupied by waves; $N^2 \equiv -\partial b/\partial z$ is the Brunt–Väisälä frequency.

Note that $\int_{-h}^{\hat{\eta}} S_{\alpha\beta}\, dz = E[(k_\alpha k_\beta/k^2)(c_g/c) + \delta_{\alpha\beta}(c_g/c - 1/2)]$ as in Phillips (1977).

3. Vertical wind stress transport

Thus far, the surface wind pressure has not been considered. However, the horizontal (kinematic) surface wind stress $\overline{\tau}_\alpha(\hat{\eta})$ can be divided into a turbulence-viscous part or skin friction $\overline{\tau}_{T\alpha}(\hat{\eta})$ and a pressure part or form drag $\overline{\tau}_{P\alpha}(\hat{\eta})$; that is,

$$\overline{\tau_\alpha}(\hat{\eta}) = \overline{\tau_{T\alpha}}(\hat{\eta}) + \overline{\tau_{P\alpha}}(\hat{\eta}), \quad \overline{\tau_{T\alpha}}(\hat{\eta}) = \overline{K_M} \left.\frac{\partial U_\alpha}{\partial z}\right|_{z=\hat{\eta}}, \quad \overline{\tau_{P\alpha}}(\hat{\eta}) = \overline{\tilde{p}_{w\tilde{\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha}}, \tag{25a), (25b), 25c}$$

where $\overline{K_M}$ is an empirical momentum mixing coefficient. The dynamic stress $\rho_w \tau_\alpha$, where ρ_w is seawater density, is continuous across the air–sea interface. The form of (25b), although conventional, is problematic and is subject to further research.

The component of the wind pressure fluctuation that correlates with $\partial\tilde{\eta}/\partial x_\alpha$ in (25c) is $\tilde{p}_{w\tilde{\eta}} = p_{w0} \sin\psi$, and its subsurface continuation is $\tilde{p}_w(z) = p_{w0} F_{\mathrm{CC}} \sin\psi$ in (1d) [which for horizontally homogeneous, deep water is implicit in a formula in Weber (1983), albeit expressed in Lagrangian coordinates]. The subsurface continuation of $\partial\tilde{\eta}/\partial x_\alpha$ is (4), so that

$$\overline{\tau_{P\alpha}}(z) = \overline{p_{w0} F_{\mathrm{CC}} \sin\psi \tilde{s}_\alpha} = \overline{p_{w0} \sin\psi \frac{\partial \tilde{\eta}}{\partial x_\alpha}} F_{\mathrm{CC}} F_{\mathrm{SS}}$$

or

$$\overline{\tau_{P\alpha}}(z) = \overline{p_{w\tilde{\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha}} F_{\mathrm{CC}} F_{\mathrm{SS}}, \tag{26}$$

and it conforms to (25c) at $z = \hat{\eta}$. The surface stress $\overline{p_{w\tilde{\eta}} \partial\tilde{\eta}/\partial x_\alpha}$ is empirical; however, the transport of this pressure stress into the water column is now a known function, unlike turbulence transport.

On sufficiently rough stationary surfaces, form drag dominates over skin friction (Schlichtng 1979), and this is assumed to prevail over wave surfaces by Smith (2006), Donelan (1999), and others for wind speeds greater than some threshold value (3 to 5 m s^{-1}). On the other hand, Janssen (1989) indicates that form drag, or "wave-induced stress," dominates only for young waves ($c_p/u_* \cong 5$, where c_p is the spectral peak phase speed and u_* is the friction velocity), while skin friction dominates for old waves ($c_p/u_* \cong 25$).

4. The wave energy equation

The wave energy equation is derived in appendix A and is

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\beta}[(c_{g\beta} + u_{A\beta})E] + \int_{-h}^{\hat{\eta}} S_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta} dz = S_W - S_{\mathrm{Diss}}. \tag{27}$$

The terms on the right must be determined empirically; the first is a wind source term and the second term is dissipation.

The energy advective velocity, as defined in appendix A, is

$$u_{A\beta} = k\int_{-h}^{\hat{\eta}} r(z)U_\beta\, dz,$$

$$r(z) = (F_{CC}F_{CS} + F_{SS}F_{SC})/2 + F_{SS}F_{CS} \quad \text{(28a), (28b)}$$

and $r(z)$ is a weight factor biasing the evaluation of $u_{A\alpha}$ toward the surface velocity and $k\int_{-h}^{\hat{\eta}} r(z)\, dz = 1$. [In deep water, $r = 2\exp 2k(z - \hat{\eta})$.]

It has been suggested that instead of using the wave energy Eq. (27), the wave action equation—conventionally used in many models—be adopted, the presumption being that the third term on the left of (27) would neatly disappear. However, because U_α is not vertically constant, the wave action equation would be insensitive to vertical profiles of U_β, unlike (27). The wave action equation is derived in appendix B with vertical velocity gradients included.

Appendix B also contains Eqs. (B.4a) and (B.5), which can be solved along with (27) and (28) to provide the intrinsic frequency and wavenumber. Alternatively for steady flow, the "encounter frequency" ω is spatially constant according to (B.2), and the simpler Eqs. (B.1), (B.3), and the dispersion relation can be used.

5. The sigma equations

The relation between the vertical Cartesian and sigma (using ς instead of σ) coordinate is

$$z = \hat{\eta} + \varsigma D. \tag{29}$$

Transforming (12) to sigma coordinates, we have

$$\frac{\partial DU_\alpha}{\partial x_\alpha} + \frac{\partial \Omega}{\partial \varsigma} + \frac{\partial \hat{\eta}}{\partial t} = 0, \tag{30}$$

and for (23)

$$\frac{\partial}{\partial t}(DU_\alpha) + \frac{\partial}{\partial x_\beta}(DU_\alpha U_\beta) + \frac{\partial}{\partial \varsigma}(\Omega U_\alpha) - \varepsilon_{\alpha\beta z} f_z DU_\beta + D\frac{\partial}{\partial x_\alpha}(g\hat{\eta} + p_{\text{atm}}) + D\int_\varsigma^0 \left(D\frac{\partial b}{\partial x_\alpha} - \varsigma\frac{\partial D}{\partial x_\alpha}\frac{\partial b}{\partial \varsigma}\right) d\varsigma =$$
$$-D\frac{\partial S_{\alpha\beta}}{\partial x_\beta} + \varsigma\frac{\partial D}{\partial x_\beta}\frac{\partial S_{\alpha\beta}}{\partial \varsigma} + \frac{\partial \overline{\tau_\alpha}}{\partial \varsigma}. \tag{31}$$

The definition of $S_{\alpha\beta}$ remains the same as in (24b), noting that $(z + h) = D(1 + \varsigma)$. The term Ω is a velocity normal to sigma surfaces and $\Omega(\hat{\eta}) = \Omega(-h) = 0$.

6. The case posed by Ardhuin et al.

As mentioned previously, this paper was stimulated by Ardhuin et al. (2008a), who cited a solution from a multimode model by Belibassakis and Athanassoulis (2002) in which currents and waves were unidirectional and propagated into a straight entry channel of 6-m depth, which smoothly transitioned to a straight exit channel of 4-m depth. Although the algorithm was complicated, the solution was simple and deemed accurate. The wave frequency was selected so that kD varied from 1.10 to 0.85, a shallow-water case; the group velocity was nearly constant and so was the wave energy (see Fig. 3.4 in Phillips 1977). They pointed out that the radiation stress terms in M03 produced a vertical gradient of mean velocity greater than zero, contrary to that of the multimode solution.

From the discussion in section 2, some error is to be expected for finite $\partial h/\partial x$. However, the results of this paper now agree with the multimode solution because, for the steady, unidirectional, irrotational case of Ardhuin et al., we have from (23)

$$\frac{\partial}{\partial x}(UU) + \frac{\partial}{\partial z}(WU_\alpha) + \frac{\partial}{\partial x}(g\hat{\eta}) = -\frac{\partial S_{xx}}{\partial x},$$

and from (24b)

$$S_{xx} = kE(F_{CS}F_{CC} - F_{SC}F_{SS}) = kE\frac{2}{\sinh 2kD},$$

because $\cosh^2 k(z + h) - \sinh^2 k(z + h) = 1$. Thus, the vertical structure vanishes as in the multimode solution; the radiation term is balanced by the hydrostatic pressure gradient (in the 6- to 4-m transition section there must be some nonhydrostatic effects). The singular term, E_D in (24b), is excluded because $\partial E/\partial x \cong 0$ in this case.

Thus, there is a good possibility that the equations in this paper do apply to shallow water for $kD \approx 1$ (where, realistically, viscous-turbulence effects should not be ignored).

7. Summary

The above results differ from M03 in several ways. Horizontal derivatives of bottom depth were retained in the M03 equivalent of (4), which is inconsistent with

the derivation of (1) based on a flat bottom. The terms $\overline{\tilde{w}^2}$ and E_D were missing in the M03 version of (24a) and a $\tilde{p} - \tilde{s}_\alpha$ correlation term erroneously substituted. The derivation and definition of U_α to include currents and Stokes drift is unchanged. An important consequence here is that for large kD, the momentum equation is not sensitive to finite $\partial h/\partial x_\alpha$ as was the M03 version. For $kD \approx 1$, some discrepancy relative to the case posed by Ardhuin et al. (2008a) is expected, but in fact the discrepancy is nil.

The terms $\overline{\tilde{u}_\alpha \tilde{u}_\beta}$, $\overline{\tilde{w}^2}$, and E_D in (24a) are, in vertically integrated form, the same as those in the derivations of Longuet-Higgins and Stewart (1962, 1964) and Phillips (1977).

Stokes drift, given by (10c), is the same as the result from the Lagrangian determination, $u_{S\alpha} = \overline{\tilde{u}_\alpha + (\partial \tilde{u}_\alpha/\partial x)\tilde{x} + (\partial \tilde{u}_\alpha/\partial z)\tilde{z}}$, where $\tilde{x} = \int \tilde{u} dt$ and $\tilde{z} = \int \tilde{w} dt$.

The basis for a coupled wave-circulation model are, in summary, provided by Eqs. (12), (23), (24), (26), (27), and (28). Empirical knowledge is needed for $\tau_{T\alpha}$, S_W and S_{Diss}. The equation for mean temperature, $\partial T/\partial t + \partial(U_\beta T)/\partial x_\beta = \partial[K_H(\partial T/\partial z)]/\partial z$—or any other scalar—appropriately uses $U_\beta = \hat{u}_\beta + u_{S\beta}$ as the advective velocity.

For a practical wave model, Eqs. (27) and (28) should be extended so that wave energy is dependent on wavenumber or frequency and wave propagation directions. Existing third-generation wave models (e.g., Tolman 1991) might be modified to conform to (27) and (28). Alternatively, a conforming, somewhat simplified wave model has been created (Mellor et al. 2008) and has since been coupled with the Princeton Ocean Model.

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APPENDIX A

Derivation of the Energy Equation

The derivation of the wave energy equation, which contains depth-independent variables, is complex. Thus, using $\partial u_k/\partial x_k = 0$, the product of u_i and $\partial u_i/\partial t + u_k \partial u_i/\partial x_k + \partial(p + gz)/\partial x_i = \partial \tau_\beta/\partial z$ is

$$\frac{\partial u_i^2/2}{\partial t} + \frac{\partial}{\partial x_\beta}[u_\beta(u_i^2/2 + p + gz)] + \frac{\partial}{\partial z}[w(u_i^2/2 + p + gz)] = u_\beta \frac{\partial \tau_\beta}{\partial z}. \tag{A.1}$$

Integrate from $z = -h$ to $z = \eta$ and use $w(\eta) = u_\beta(\eta)\partial\eta/\partial x_\beta + \partial\eta/\partial t$ and $w(-h) = -u_\beta(-h)\partial h/\partial x_\beta$. Assuming that $\partial\hat{\eta}/\partial t \ll \partial\tilde{\eta}/\partial t$, we have after phase averaging

$$\frac{\partial}{\partial t}\left[\overline{\int_{-h}^{\eta} \frac{u_i^2}{2} dz} + g\frac{\overline{\tilde{\eta}^2}}{2}\right] + \frac{\partial}{\partial x_\beta}\overline{\int_{-h}^{\eta} u_\beta\left(\frac{u_i^2}{2} + gz + p\right) dz} = \overline{\int_{-h}^{\eta} u_\beta \frac{\partial \tau_\beta}{\partial z} dz} - \overline{p_{w\eta}\frac{\partial \tilde{\eta}}{\partial t}}. \tag{A.2}$$

The surface wind pressure is $p_{w\eta}$ as in section 3, and the last term on the right is the rate of work done by wind pressure.

It should be noted that velocity terms on the left of (A.2) are dominated by the wave components and that this discussion could be shortened by simply discarding current and Stokes contributions. Alternately, the integrals could be divided into integrals from $-h$ to $\hat{\eta}$ and $\hat{\eta}$ to η (Phillips 1977); evaluating the velocity terms proved complicated. As a reasonable approximation, replace the velocity integrals in (A.2) so that

$$\frac{\partial}{\partial t}\left[\int_{-h}^{\hat{\eta}} \overline{\frac{u_i^2}{2} s_z} dz + g\frac{\overline{\tilde{\eta}^2}}{2}\right] + \frac{\partial}{\partial x_\beta}\left[\int_{-h}^{\hat{\eta}} \overline{u_\beta s_z\left(\frac{u_i^2}{2} + gs\right)} dz + \int_{-h}^{\hat{\eta}} \overline{u_\beta p\, dz} + \overline{\int_{\hat{\eta}}^{\eta} u_\beta g(\eta - z)\, dz}\right] = \int_{-h}^{\hat{\eta}} \overline{u_\beta \frac{\partial \tau_\beta}{\partial z}} dz - \overline{p_{w\eta}\frac{\partial \tilde{\eta}}{\partial t}}. \tag{A.3}$$

Notice that $\int_{-h}^{\eta} dz = \int_{-h}^{\hat{\eta}} s_z\, dz = \hat{\eta} + \tilde{\eta} + h$, so that the role of s_z is to span the entire vertical range even though the upper limit has been changed. Because the pressure terms differ analytically in the regions $-h < z < \hat{\eta}$ and

$-|\tilde{\eta}| < z - \hat{\eta} < |\tilde{\eta}|$ as given by (19), the pressure integrals are similarly divided.

Next, evaluate the terms in (A.3). Thus,

$$\frac{\overline{u_i^2 s_z}}{2} = \frac{\hat{u}_\alpha^2}{2} + \hat{u}_\alpha \overline{u_{e\alpha} s_z} + \frac{\overline{\tilde{u}_{ei}^2}}{2} = \frac{\hat{u}_\alpha^2}{2} + \hat{u}_\alpha u_{S\alpha} + \frac{\overline{\tilde{u}_i^2}}{2}.$$

After substituting $\hat{u}_\alpha = U_\alpha - u_{S\alpha}$,

$$\frac{\overline{\tilde{u}_i^2 s_z}}{2} = \frac{U_\alpha^2}{2} + \frac{\overline{\tilde{u}_i^2}}{2}. \tag{A.4a}$$

In the above, a term of order $(ka)^4$ has been expunged. Similarly,

$$\overline{u_\beta \frac{u_i^2}{2} s_z} = U_\beta\left(\frac{U_\alpha^2}{2} + \frac{\overline{\tilde{u}_\alpha^2}}{2}\right) + U_\alpha \overline{\tilde{u}_\alpha \tilde{u}_\beta} \tag{A.4b}$$

and

$$\overline{u_\beta p} = (\hat{u}_\beta + \overline{\tilde{u}_{e\beta} s_z})\hat{p} + \overline{\tilde{u}_{e\beta}\tilde{p}} = U_\beta \hat{p} + \overline{\tilde{u}_\beta \tilde{p}},$$

where $(\hat{u}_\beta + \overline{\tilde{u}_{e\beta} s_z}) \cong U_\beta$, so that using (18),

$$\overline{u_\beta p} = U_\beta(-\overline{\tilde{w}^2} + g(\hat{\eta} - z) + p_{\mathrm{atm}}) + \overline{\tilde{u}_\beta \tilde{p}}. \tag{A.4c}$$

Finally,

$$\int_{\hat{\eta}}^{\hat{\eta}+\tilde{\eta}} \overline{u_\beta g(\eta - z)}\, dz = \hat{u}_\beta E_D \cong U_\beta E_D. \tag{A.4d}$$

Substituting $U_\beta E_D$ for $\hat{u}_\beta E_D$ introduces an error of order $(ka)^4$.

Inserting (A.4) into (A.3) yields

$$\frac{\partial}{\partial t}\left[\int_{-h}^{\hat{\eta}}\left(\frac{U_\alpha^2}{2} + \frac{\overline{\tilde{u}_i^2}}{2}\right) dz + g\frac{\overline{\tilde{\eta}^2}}{2}\right] + \frac{\partial}{\partial x_\beta}\int_{-h}^{\hat{\eta}} U_\beta\left(\frac{U_\alpha^2}{2} + \frac{\overline{\tilde{u}_i^2}}{2}\right) dz$$
$$+ \frac{\partial}{\partial x_\beta}\int_{-h}^{\hat{\eta}} [U_\beta(g\hat{\eta} + g\overline{\tilde{s}\tilde{s}_z} + p_{\mathrm{atm}}) + U_\alpha S_{\alpha\beta} + \overline{\tilde{u}_\beta \tilde{p}}]\, dz = \int_{-h}^{\hat{\eta}} \overline{u_\beta \frac{\partial \tau_\beta}{\partial z}}\, dz - \overline{p_{w\eta}\frac{\partial \tilde{\eta}}{\partial t}}, \tag{A.5}$$

where $g(\overline{ss_z} - z) = g\overline{\tilde{s}\tilde{s}_z}$ and $S_{\alpha\beta}$ is defined in (24).

Next, the product of U_α and (23) yields a mean flow (current plus Stokes) energy equation:

$$\frac{\partial}{\partial t}\int_{-h}^{\hat{\eta}} \frac{U_\alpha^2}{2} dz + \frac{\partial}{\partial x_\beta}\int_{-h}^{\hat{\eta}} U_\beta \frac{U_\alpha^2}{2} dz + \frac{\partial}{\partial x_\alpha}\int_{-h}^{\hat{\eta}} U_\alpha(g\hat{\eta} + p_{\mathrm{atm}})\, dz + \int_{-h}^{\hat{\eta}} U_\alpha \frac{\partial S_{\alpha\beta}}{\partial x_\beta} dz = \int_{-h}^{\hat{\eta}} U_\beta \frac{\partial \overline{\tau}_\beta}{\partial z} dz. \tag{A.6}$$

Subtracting (A.6) from (A.5) gives

$$\frac{\partial}{\partial t}\left[\int_{-h}^{\hat{\eta}} \frac{\overline{\tilde{u}_i^2}}{2} dz + g\frac{\overline{\tilde{\eta}^2}}{2}\right] + \frac{\partial}{\partial x_\beta}\left[\int_{-h}^{\hat{\eta}} U_\beta\left(\frac{\overline{\tilde{u}_i^2}}{2} + \overline{\tilde{s}\tilde{s}_z}\right) dz + \int_{-h}^{\hat{\eta}} S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} dz + \int_{-h}^{\hat{\eta}} \overline{\tilde{u}_\beta \tilde{p}}\, dz\right] = -\overline{p_{w\eta}\frac{\partial \tilde{\eta}}{\partial t}}$$
$$+ \int_{-h}^{\hat{\eta}} \overline{u_\beta \frac{\partial \tau_\beta}{\partial z}}\, dz - \int_{-h}^{\hat{\eta}} U_\beta \frac{\partial \overline{\tau}_\beta}{\partial z} dz. \tag{A.7}$$

The wave energy emerges as

$$E = \int_{-h}^{\hat{\eta}} \frac{\overline{\tilde{u}_i^2}}{2} dz + g\frac{\overline{\tilde{\eta}^2}}{2}. \tag{A.8}$$

The two terms on the right are equal and $E = ga^2/2$.

The pressure–velocity correlation is important because

$$\int_{-h}^{\hat{\eta}} \overline{\tilde{u}_\beta \tilde{p}}\, dz = k\int_{-h}^{\hat{\eta}} F_{CS}F_{CC}\, dz\left(\frac{ck_\beta}{k}\frac{ga^2}{2}\right) = c_{g\beta}E, \tag{A.9a}$$

where $c_g = (c/2)(1 + 2kD/\sinh 2kD)$ is the group speed and $c_{g\beta} = k_\beta c_g/k$. An energy advective velocity is defined such that

$$u_{A\beta} = E^{-1}\int_{-h}^{\hat{\eta}} U_\beta\left(\frac{\overline{\tilde{u}_i^2}}{2} + \overline{\tilde{s}\tilde{s}_z}\right) dz$$

or

$$u_{A\beta} = k\int_{-h}^{\hat{\eta}} U_\beta[(F_{CC}F_{CS} + F_{SS}F_{SC})/2 + F_{SS}F_{CS}]\, dz \tag{A.9b}$$

as in (28). The terms in square brackets integrate to unity so that it is a weighting factor; in deep water it selects the near-surface wave portion of U_β as contributions to $u_{A\beta}$.

Inserting (A.8) and (A.9) into (A.7) yields

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\beta}[(c_{g\beta} + u_{A\beta})E] + \int_{-h}^{\hat{\eta}} S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}\,dz = S_W - S_{\mathrm{Diss}} \tag{A.10}$$

as in (27). Here, S_W is the wind energy source defined below and S_{Diss} is dissipation.

Wind energy source

Recall that in (25) the total surface stress has two parts, the turbulence part given by (25b) and the pressure part given by (25c). For the latter, $\overline{u_\beta \partial\tau_{P\beta}/\partial z} = \hat{u}_\beta \partial\overline{\tau_{P\beta}}/\partial z$, to which we add the term $\overline{p(\tilde{\eta})\partial\tilde{\eta}/\partial t}$ from (A.7). Recalling that $U_\beta = \hat{u}_\beta + u_{S\beta}$, we have

$$-\overline{p_{w\eta}\frac{\partial\tilde{\eta}}{\partial t}} - \int_{-h}^{\hat{\eta}} u_{S\beta}\frac{\partial\overline{\tau_{P\beta}}}{\partial z}\,dz = \overline{p_{w\eta}\frac{\partial\tilde{\eta}}{\partial x_\beta}}\left(c_\beta + \frac{k_\alpha k_\beta}{k^2}\hat{U}_\alpha - \int_{-h}^{\hat{\eta}} u_{S\beta}(F_{\mathrm{SC}}F_{\mathrm{SS}} + F_{\mathrm{CC}}F_{\mathrm{CS}})k\,dz\right), \tag{A.11}$$

where we have used $\partial\tilde{\eta}/\partial t = -(\omega k_\beta/k^2)\partial\tilde{\eta}/\partial x_\beta$ and $\omega = \sigma + k_\alpha\hat{U}_\alpha$. The last term in (A.11) is order $(ka)^2$ smaller than the phase speed, and the wave energy forcing can be represented by $[c_\beta + (k_\alpha k_\beta/k^2)\hat{U}_\alpha]\overline{p_{w\eta}\partial\tilde{\eta}/\partial x_\beta}$. Therefore,

$$S_W = \left(c_\beta + \frac{k_\alpha k_\beta}{k^2}\hat{U}_\beta\right)\overline{p_{w\eta}\frac{\partial\tilde{\eta}}{\partial x_\beta}}, \tag{A.12}$$

and, presumably

$$S_{\mathrm{Diss}} = -\int_{-h}^{\hat{\eta}} \overline{(\tilde{u}_\beta - u_{S\beta})\frac{\partial\tau_{T\beta}}{\partial z}}\,dz. \tag{A.13}$$

Generally, $|\hat{U}_\beta| \ll |c_\beta|$; otherwise, $\hat{U}_\beta = u_{A\beta}$ should be a good approximation.

APPENDIX B

Derivation of the Wave Action Equation

a. Wave kinematics

The relation between absolute frequency ω, intrinsic frequency σ, wavenumber vector $k_\beta = (k_x, k_y)$, and the Doppler velocity $\hat{U}_\alpha$, is

$$\omega = \sigma + k_\alpha\hat{U}_\alpha. \tag{B.1}$$

From the definition of phase, $\psi = k_\alpha x_\alpha - \omega t$,

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial\omega}{\partial x_\alpha} = 0, \tag{B.2}$$

$$\frac{\partial k_\alpha}{\partial x_\beta} - \frac{\partial k_\beta}{\partial x_\alpha} = 0. \tag{B.3}$$

Using (B.1) in (B.2), where $\sigma = \sigma(k, D)$, the relations $\partial k/\partial x_\alpha = (k_\beta/k)\partial k_\beta/\partial x_\alpha$, $c_{g\beta} = (k_\beta/k)(\partial\sigma/\partial k)$, and (B.3), one obtains a wavenumber equation:

$$\frac{\partial k_\alpha}{\partial t} + (c_{g\beta} + \hat{U}_\beta)\frac{\partial k_\alpha}{\partial x_\beta} = -\frac{\partial\sigma}{\partial D}\frac{\partial D}{\partial x_\alpha} - k_\beta\frac{\partial\hat{U}_\beta}{\partial x_\alpha} \tag{B.4a}$$

or

$$\frac{\partial k}{\partial t} + (c_{g\beta} + \hat{U}_\beta)\frac{\partial k}{\partial x_\beta} = -\frac{k_\alpha}{k}\frac{\partial\sigma}{\partial D}\frac{\partial D}{\partial x_\alpha} - \frac{k_\alpha k_\beta}{k}\frac{\partial\hat{U}_\beta}{\partial x_\alpha}. \tag{B.4b}$$

Forming $\partial\sigma/\partial t + c_{g\beta}\partial\sigma/\partial x_\beta$, using $\sigma = \sigma(k, D)$ again and (B.4b) and canceling two equal but opposite terms containing $\partial D/\partial x_\beta$, yields the intrinsic frequency equation

$$\frac{\partial\sigma}{\partial t} + (c_{g\beta} + \hat{U}_\beta)\frac{\partial\sigma}{\partial x_\beta} = -\frac{\partial\sigma}{\partial k}\left[\frac{k_\alpha k_\beta}{k}\frac{\partial\hat{U}_\beta}{\partial x_\alpha}\right] + \frac{\partial\sigma}{\partial D}\left[\frac{\partial D}{\partial t} + \hat{U}_\beta\frac{\partial D}{\partial x_\beta}\right]. \tag{B.5}$$

When specialized to depth-independent currents, (B.4b) and (B.5) are the same as those in Bretherton and Garrett (1969, their appendix) and Tolman (1991).

From the dispersion relation, $\sigma^2 = gk\tanh kD$, $\partial\sigma/\partial D = (\sigma/D)(c_g/c - 1/2)$, and from the continuity equation,

$$\frac{\partial D}{\partial t} + \overline{U}_\beta\frac{\partial D}{\partial x_\beta} = -D\frac{\partial\overline{U}_\beta}{\partial x_\beta}, \tag{B.6}$$

where

$$\overline{U}_\alpha \equiv \frac{1}{D}\int_{-h}^{\eta} U_\alpha(z)\,dz \tag{B.7}$$

is the vertically averaged current. Inserting (B.6) into (B.5), one obtains

$$\frac{\partial\sigma}{\partial t} + (c_{g\beta} + \hat{U}_\beta)\frac{\partial\sigma}{\partial x_\beta} = -\sigma\frac{c_g}{c}\frac{k_\alpha k_\beta}{k^2}\frac{\partial\hat{U}_\beta}{\partial x_\alpha} - \sigma\left(\frac{c_g}{c} - \frac{1}{2}\right)\left(\frac{\partial\overline{U}_\beta}{\partial x_\beta} + \frac{\overline{U}_\beta - \hat{U}_\beta}{D}\frac{\partial D}{\partial x_\beta}\right). \tag{B.8}$$

b. The wave energy and action equations

Excluding the right side of (27) or (A.10), we have

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\beta}[(c_{g\beta} + u_{A\beta})E] = -\frac{1}{D}\int_{-h}^{\eta} S_{\alpha\beta}\frac{\partial U_\beta}{\partial x_\alpha}\,dz. \tag{B.9}$$

The wave action equation is

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \frac{\partial}{\partial x_\beta}\left[(c_{g\beta} + u_{A\beta})\frac{E}{\sigma}\right] = \frac{1}{\sigma}\frac{\partial E}{\partial t} - \frac{E}{\sigma^2}\frac{\partial \sigma}{\partial t} + \frac{1}{\sigma}\frac{\partial}{\partial x_\alpha}[(c_{g\alpha} + u_{A\beta})E] - \frac{E}{\sigma^2}(c_{g\beta} + u_{A\beta})\frac{\partial \sigma}{\partial x_\beta}. \tag{B.10}$$

Substituting (B.8) and (B.9) into (B.10), one has

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \frac{\partial}{\partial x_\beta}\left[(c_{g\beta} + u_{A\beta})\frac{E}{\sigma}\right] = -\frac{1}{\sigma D}\int_{-h}^{\eta} S_{\alpha\beta}\frac{\partial U_\beta}{\partial x_\alpha}\,dz + \frac{E}{\sigma}\frac{c_g}{c}\frac{k_\alpha k_\beta}{k^2}\frac{\partial \hat{U}_\beta}{\partial x_\alpha} + \frac{E}{\sigma}\left(\frac{c_g}{c} - \frac{1}{2}\right)\left(\frac{\overline{U}_\beta - \hat{U}_\beta}{D}\frac{\partial D}{\partial x_\beta} + \frac{\partial \overline{U}_\beta}{\partial x_\beta}\right). \tag{B.11}$$

We have approximated $\hat{U}_\beta = u_{A\beta}$ in the left side of (B8); otherwise, the final result in (B.11) will contain additional terms proportional to $\hat{U}_\beta - u_{A\beta}$.

Now an integral of (24b) (Phillips 1977) is

$$\frac{1}{\sigma D}\int_{-h}^{\eta} S_{\alpha\beta}\,dz = \frac{E}{\sigma}\frac{c_g}{c}\frac{k_\alpha k_\beta}{k^2} + \frac{E}{\sigma}\left(\frac{c_g}{c} - \frac{1}{2}\right)\delta_{\alpha\beta}. \tag{B.12}$$

If $U_\beta(z)$ is vertically constant and equal to $\overline{U}_\beta$ and $\hat{U}_\beta$, then the horizontal current gradient in the first term on the right of (B.11) can be taken outside of the integral, so that all of the terms on the right of (13) cancel (Mei 1983) and one obtains the conventional wave action equation. But, generally, in a three-dimensional ocean, $U_\beta(z) \neq \hat{U}_\beta \neq \overline{U}_\beta$.

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