The Gaussian Cloud Model Relations[1]

GEORGE L. MELLOR

Geophysical Fluid Dynamics Program, Princeton University, Princeton, N. J. 08540

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ABSTRACT

Sommeria and Deardorff (1977) have derived turbulence closure relations which should be important to cloud modeling. To obtain these relations they have hade to invoke some analytical approximations and data from numerical statistical experiments. In the present paper, the analytical approximations have been eliminated. Somewhat surprisingly, results obtained here agree exactly with those obtained by the previous authors. Other new and useful relations are presented.

1. Introduction

In the immediately preceding paper Sommeria and Deardorff (1977, henceforth SD) have derived relations which should have major significance for both subgrid-scale modeling and ensemble-mean modeling of clouds in the atmospheric boundary layer. A Gaussian distribution of the conserved variables, liquid water potential temperature and total water specific humidity was assumed [and therefore differs from work by Manton and Cotton (1976) where, otherwise, some of the basic ideas are established] and relations necessary to close models which include clouds were derived. In their derivation, analytical approximations and numerical statistical experiments were invoked to establish the final result which presumably would also be approximate.

When the present writer first saw the Sommeria and Deardorff manuscript, he had been working along similar lines in an attempt to extend the Princeton ensemble-mean or second-moment turbulence model[2] (Mellor, 1973; Mellor and Yamada, 1974; Yamada and Mellor, 1975) so that it would properly account for clouds. Much of this work was anticipated by SD. However, a way had been found to eliminate the aforementioned analytical approximations and this is presented here; somewhat surprisingly the final result is identical to that obtained by SD. Their findings are therefore found to be exact (approximations invoked by them apparently introduce errors which are exactly self-cancelling) and methodology is established which may be useful in the investigation of non-Gaussian distributions, consideration of which may further improve ensemble-mean models, for example. Other useful relations are also obtained here.

It should be noted that, whereas these relations can improve subgrid-scale modeling which had previously demonstrated impressive predictive power (e.g., Deardorff, 1973; Sommeria, 1976), the relations are probably more essential to the inclusion of clouds in ensemble-mean models. The latter, of course, are more dependent on empirical modeling than the former but require much less resolution in space and time such that they can be included in current general circulation models. In such an application a probability distribution of liquid water, for example, would represent the probability of the spacial distribution in a horizontal grid cell of the order of (100 km)^2.

2. Analysis

We adopt the nomenclature of SD. We note that it is a straightforward extension of previous dry models to write model equations for the mean and variances of the conserved variables, liquid water potential temperature

$$\theta_l \equiv \theta - \frac{\bar{\theta}}{\bar{T}}\frac{L}{c_p}q_l \tag{1}$$

and total water specific humidity

$$q_w \equiv q + q_l, \tag{2}$$

where, neglecting pressure fluctuations,

$$\theta \equiv \left(\frac{p_0}{p}\right)^{0.285} T = \frac{\bar{\theta}}{\bar{T}}T \tag{3}$$

is the potential temperature, T the absolute temperature, q_l the liquid-water specific humidity, q the water vapor specific humidity, q_w the total water specific

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[2] Unfortunately, the term "second-order model" has gained more acceptance.

humidity, L the latent heat of vaporization and c_p the specific heat at constant pressure. The means of each quantity are denoted by overbars such that $\bar{q}_w$ is the mean total water specific humidity and primes denote fluctuating quantities such as $q_w'=q_w-\bar{q}_w$. As stated previously, it is presumed that predictive equations are available for $\bar{q}_w$, $\overline{q_w'^2}$, $\bar{\theta}_l$, $\overline{\theta_l'^2}$ and $\overline{\theta_l' q_w'}$. We require $\bar{q}_l$ and other useful statistical quantities involving q_l' to be discussed below.

The detailed, local condensation physics is assumed to be given by

$$q_l=(q_w-q_s)H(q_w-q_s), \tag{4}$$

where q_s is the saturation specific humidity and

$$H(x)\equiv\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$$

is a Heaviside function. Thus this "fast condensation" model assumes no supersaturation; when $q_w>q_s$, $q=q_s$. Note that, were it not for the Heaviside function, that is, if $q_l=q_w-q_s$, then we would have $\bar{q}_l=\bar{q}_w-\bar{q}_s$.

Following SD, we now assume a Gaussian or binormal distribution for θ_l and q_w, i.e.,

$$G=\frac{1}{2\pi\sigma_{\theta l}\sigma_{qw}(1-r^2)^{\frac{1}{2}}}\times\exp\left[\frac{-1}{1-r^2}\left\{\frac{\theta_l'^2}{2\sigma_{\theta l}^2}-r\frac{\theta_l' q_w'}{\sigma_{\theta l}\sigma_{qw}}+\frac{q_w'^2}{2\sigma_{qw}^2}\right\}\right], \tag{5}$$

where $\sigma_{\theta l}^2\equiv\overline{\theta_l'^2}$, $\sigma_{qw}^2\equiv\overline{q_w'^2}$ and $r\equiv\overline{\theta_l' q_w'}/\sigma_{\theta l}\sigma_{qw}$. We now seek expressions for the cloud fraction

$$R=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(q_w-q_s)G\,dq_w d\theta_l \tag{6}$$

and the mean liquid water specific humidity

$$\bar{q}_l=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q_w-q_s)H(q_w-q_s)G\,dq_w d\theta_l. \tag{7}$$

With the help of a truncated Taylor series and (1) and (3), it is possible to write

$$q_s(T)=\bar{q}_{sl}+\bar{q}_{sl,T}[T-\bar{T}_l]=\bar{q}_{sl}+\bar{q}_{sl,T}\left[\frac{\bar{T}}{\bar{\theta}}\theta_l'+\frac{L}{c_p}q_l\right], \tag{8}$$

where

$$\bar{q}_{sl}\equiv q_s(\bar{T}_l),\quad \bar{T}_l\equiv\frac{\bar{T}}{\bar{\theta}}\bar{\theta}_l, \tag{9a,b}$$

and

$$\bar{q}_{sl,T}\equiv\left(\frac{\partial q_s}{\partial T}\right)_{T=\bar{T}_l}=0.622\frac{L\bar{q}_{sl}}{R_d\bar{T}_l^2}, \tag{9c}$$

where the latter is the Clasius-Clapeyron equation. Now for $q_w>q_s$ and $q_l=q_w-q_s$ we obtain

$$q_w-q_s=a\Delta\bar{q}+aq_w'-b\theta_l' \tag{10}$$

from (8) after some algebra, where

$$a\equiv\left(1+\bar{q}_{sl,T}\frac{L}{c_p}\right)^{-1}, \tag{11a}$$

$$b\equiv a\frac{\bar{T}}{\bar{\theta}}q_{sl,T}, \tag{11b}$$

$$\Delta\bar{q}\equiv\bar{q}_w-\bar{q}_{sl}. \tag{11c}$$

When (10) is inserted into (6) and (7) the following transformations are suggested:

$$aq_w'=t+s,\quad b\theta_l'=t-s \tag{12a,b}$$

or conversely

$$t=(aq_w'+b\theta_l')/2,\quad s=(aq_w'-b\theta_l')/2.$$

Therefore, (6) and (7) may be written

$$R=2\int_{-a\Delta\bar{q}/2}^{\infty}\int_{-\infty}^{\infty}\tilde{G}(s,t)dtds, \tag{13}$$

$$\bar{q}_l=2\int_{-a\Delta\bar{q}/2}^{\infty}\int_{-\infty}^{\infty}a\Delta\bar{q}+2s)\tilde{G}(s,t)dtds, \tag{14}$$

where $\tilde{G}(s,t)$ results from substituting (12a,b) into Eq. (5).[3] The integration with respect to t may be executed to give

$$\tilde{G}(s)\equiv2\int_{-\infty}^{\infty}\tilde{G}(s,t)dt=\frac{1}{\sqrt{2\pi}\sigma_s}\exp\left[-\frac{s^2}{2\sigma_s^2}\right], \tag{15}$$

where

$$\sigma_s^2\equiv\overline{s^2}=\tfrac{1}{4}(a^2\overline{q_w'^2}-2ab\overline{q_w'\theta_l'}+b^2\overline{\theta_l'^2}). \tag{16}$$

Thus, we obtain the comfortable result that a variable comprised of a linear combination of two other binormally distributed variables is also normally distributed.

We are finally left with

$$\left.\begin{aligned}R&=\int_{-a\Delta\bar{q}/2}^{\infty}\tilde{G}(s)ds\\ \bar{q}_l&=\int_{-a\Delta\bar{q}/2}^{\infty}(a\Delta\bar{q}+2s)\tilde{G}(s)ds\end{aligned}\right\},$$

[3] This yields

$$\tilde{G}(s,t)=[2\pi ab\sigma_{\theta l}\sigma_{qw}(1-r^2)^{\frac{1}{2}}]^{-1}\exp[-At^2-Bt-C],$$

where

$$A\equiv(1-r^2)^{-1}[(2a^2\sigma_{qw}^2)^{-1}-r(ab\sigma_{qw}\sigma_{\theta l})^{-1}+(2b^2\sigma_{\theta l}^2)^{-1}],$$
$$B\equiv(1-r^2)^{-1}[(a^2\sigma_{qw}^2)^{-1}-(b^2\sigma_{\theta l}^2)^{-1}],$$
$$C\equiv(1-r^2)^{-1}[(2a^2\sigma_{qw}^2)^{-1}+r(ab\sigma_{qw}\sigma_{\theta l})^{-1}+(2b^2\sigma_{\theta l}^2)^{-1}].$$

which may now be integrated to give

$$R=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{Q_1}{\sqrt{2}}\right)\right], \tag{17}$$

$$\frac{\bar{q}_l}{2\sigma_s}=RQ_1+\frac{1}{2\pi}\exp\left[-\frac{Q_1^2}{2}\right], \tag{18}$$

where

$$Q_1\equiv\frac{a\Delta\bar{q}}{2\sigma_s}. \tag{19}$$

We now note that the above expressions are identical to the SD Eqs. (16) and (20) (the former equation obtained with the help of approximations which apparently introduce self-cancelling errors) if we observe that $a=\lambda_1$ and $4\sigma_s^2/a^2=\sigma_1^2$.

3. Further closure relations

For subgrid-scale models or ensemble mean models it is also necessary to obtain relations for $\overline{\theta_l'q_l'}$, $\overline{q_w'q_l'}$, and $\overline{u_i'q_l}$ where $u_i'=(u',v',w')$. The first two quantities are obtained as a rather straightforward extension of the previous analysis. The results are

$$b\overline{q_l'\theta_l'}=-a\overline{q_l'q_w'}=4\sigma_s^2R. \tag{20}$$

We also add the liquid water variance

$$\frac{\overline{q_l'^2}}{4\sigma_s^2}=[2+Q_1^2]R+\frac{3}{\sqrt{2\pi}}Q_1\exp\left[-\frac{Q_1^2}{2}\right] \tag{21}$$

as a possibly useful additional quantity.

To obtain $\overline{u_i'q_l'}$ it is necessary to assume that u_i' is also normally distributed such that

$$\tilde{G}(s,u_i')=\frac{1}{2\pi\sigma_s\sigma_{ui}(1-r_{sui}^2)^{\frac{1}{2}}}$$

$$\times\exp\left[\frac{-1}{1-r_{sui}^2}\left\{\frac{s^2}{2\sigma_s^2}-r\frac{su_i'}{\sigma_s\sigma_{ui}}+\frac{u_i'^2}{2\sigma_{ui}^2}\right\}\right], \tag{22}$$

where

$$\sigma_{ui}\equiv\overline{u_i'^2}\ \ [\text{no summation on } i], \tag{23a}$$

$$r_{sui}=\frac{\overline{su_i}}{\sigma_s\sigma_{ui}}=\frac{1}{2\sigma_s\sigma_{ua}}(a\overline{q_w'u_i'}-b\overline{\theta_l'u_i'}). \tag{23b}$$

The result of carrying out the integration

$$\overline{u_i'q_l'}=\int_{-a\Delta\bar{q}/2}^{\infty}\int_{-\infty}^{\infty}(a\Delta\bar{q}+2s)u_i'\tilde{G}(s,u_i')du_i ds$$

is

$$\overline{u_i'q_l'}=(a\overline{u_i'q_w'}-b\overline{u_i'\theta_l'})R. \tag{24}$$

4. Discussion

Although (20) and (21) were not obtained by SD—and the former relation should be required by all models—it appears that (24) has also been anticipated by them. It will be noted that (24) is a rather simple interpolation between the dry limit $q_l'=0$ when $R=0$ and the asymptotic moist limit $q_l'=aq_w'-b\theta_l'$ when $R\rightarrow 1$. This appears to be the nature of the assumption made by SD.

Although major analytical approximations have been eliminated, an *a priori* source of error can be identified with the fact that a Gaussian distribution for θ_l and q_w yields a finite probability that they be negative. However, this is not a practical problem so long as $\overline{\theta_l'^2}/\bar{\theta}_l^2$ and $\overline{q_w'^2}/\bar{q}_w^2$ are small. In this connection one notes that in the limit, $\bar{q}_w=\overline{q_w'^2}=0$ the cloud fraction given by (17) does not limit identically to zero. However, assuming typical values for $\Delta\bar{q}=-\bar{q}_{sl}$ and $\overline{\theta_l'^2}$, one finds that R is extremely small. Nevertheless in the application of this type of approach to ensemble mean models, probability distributions other than Gaussian should be examined. On the other hand, use of the present relations should undoubtedly be superior to models which neglect variances in q_l and which necessarily must require that $\bar{q}_l=\bar{q}_w-\bar{q}_s$.

Such models would predict either clear sky or stratus. Aside from dynamical effects of including the terms $\overline{q_l'\theta_l}$, $\overline{q_l'q_w}$ and $\overline{u_i'q_l}$ in the dynamical turbulent moment equations, it should be possible to relate cloud types to R, $\bar{q}_l$ and $\overline{q_l'^2}$. For example, stratus would imply $\bar{q}_l>0$ and $R\approx 1.0$, whereas scattered cumuli would imply $\bar{q}_l>0$ but $R\ll 1.0$; alternately, the use of $\bar{q}_l$ and $\overline{q_l'^2}$ might be convenient variables to quantitatively classify clouds.

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