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A mean motion model of the general circulation

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Summary

Equations are constructed to represent quasi-stationary mean flow of momentum and heat on a spherical earth, averaged over a long period of time such as a year and over latitude circles. The crucial shearing Reynolds stress associated with meridional transfer of zonal velocity is assumed to depend linearly on a product of the earth's angular velocity, Ω, and the meridional gradient of mean temperature; the shearing stresses associated with vertical transfer of zonal velocity and of meridional velocity are assumed to depend linearly on the vertical gradients of zonal and of meridional mean velocities respectively, and the mean eddy transfer of heat along a meridian is assumed to depend linearly on the mean meridional temperature gradient. All proportionality coefficients are taken to be independent of latitude. Two forms are assumed for the non-adiabatic atmospheric heat source function, Q, used in the thermodynamic equation. In the first case Q is assumed known (from analyses of observations) as a function of height and latitude. In the second case, Q incorporates a heating term which is partly controlled by the model itself and represents some of the characteristics of sensible and latent heat transfer. A solution of the basic equations is obtained in both cases in the form of double expansions in powers of two parameters, one depending on Ω and the other on ΔT, the mean annual temperature difference between equator and pole. The solution is evaluated using Fourier techniques.

The series expansions are found to be reasonably convergent for realistic values of the various parameters involved, three terms only being required in the ΔT expansion and five terms at most in the Ω expansion, but extensive numerical evaluation by digital computer is involved: the region considered is bounded by the tropopause and lies between the equator and 70° latitude. The computed zonal velocity has the characteristic east-west variation with latitude and a broad band maximum of 19 m sec^{-1} and the meridional velocity the characteristic tricellular structure. A poleward eddy angular momentum flux and polar inversion are predicted.

The results, through verification of the postulates, add support to the Rossby view of the general circulation in which the cyclonic-scale eddies act to release potential energy of the atmosphere to supply their own kinetic energy and form the mean zonal kinetic energy. They further indicate the value of the reconstructed 'austausch' approach for this problem.

1. Introduction

Using forcing functions for density and temperature effects, Davies and Oakes (1962) showed that a series solution can be obtained for equations describing quasi-stationary mean flow of momentum on a spherical earth. The terms in their equations represent averages over a long period of time, such as a year, and over circles of latitude and the solution was obtained in powers of a parameter depending on the earth's angular velocity, Ω. Of course in the actual atmosphere there are differences in some of the climatic features from one year to another, but the main characteristics of momentum and heat flow averaged over a year change only very slowly from one year to the next. It therefore appears to be a valid approach to attempt the construction of a stationary model; it may be possible that time variations can be considered by a similar technique at a later stage.

In recent years numerical integration of the primitive equations describing the large-scale atmospheric flow has successfully reproduced most of the characteristics of the general circulation and provided information on the various transformations taking place within the system. However, the results produced by such integrations for a given initial state are dependent upon the numerical procedure employed. This inconsistency and the stability criterion make the integration procedure prohibitive at present for discussing the longer term climatological variations. To discuss the long-term evolutions a turbulence

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theory must be developed in which some of the main phenomena present in the numerical integrations are represented by some simpler statistical mean relationship. A step towards this was made by Smagorinsky (1964) in the Symons Memorial lecture when he formulated quasi-stationary mean flow equations in terms of the eddy heat and momentum fluxes. The success of this original approach was possible because of the secondary nature of the mean variables. A significant consequence but drawback of this method was that the mean zonal wind could not be derived from the results.

The failure of the classical austausch approach, the use of which Smagorinsky (1964) circumvented, was due to the incorrect formulation of the Reynolds stress terms and to using an eddy viscosity to replace molecular viscosity, a concept particularly erroneous for the Reynolds stress associated with the meridional transfer of angular momentum. Instead, it is essential to relate the Reynolds correlations to the responsible mechanisms. As the baroclinicity of the atmosphere and not the velocity shear is responsible for this lateral mixing, Davies and Oakes (1962) assumed that the lateral stress was proportional to the meridional temperature gradient. They did not, however, produce any theory corresponding to the mixing length theory for eddy viscosity to support this postulate. Following Smagorinsky's (1964) formulation of the eddy stresses it is possible to relate the eddy transfer coefficients to the scale parameters of the flow, thus bringing the unusual $\overline{u' v'}$ hypothesis into a more definite framework.

2. Formulation of basic dynamical equations

The geometrical form of the earth is not, of course, an exact sphere but the central structure of the large-scale flow, which we wish to investigate, is driven by baroclinic disturbances and it is not likely to be influenced significantly by departures from exact sphericity. The analysis can be modified to include the usual meteorological approximation of omitting the Ω^2 term. Taking r and θ to denote distance from the centre of the spherical shell of air and colatitude respectively, denoting instantaneous meridional, zonal and vertical velocity components by V, U, W, pressure by p, density by ρ, and gravitational attraction on unit mass near the earth's surface by g (as its value is effectively equal to the resultant gravity) and neglecting all viscous stresses since the Reynolds stresses appearing finally in the equations of mean motion are very much larger, the instantaneous dynamical equations can be conveniently written in polar co-ordinate form. We obtain

$$\rho\left(\frac{\partial V}{\partial t}+W\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial V}{\partial \theta}-\frac{U^2\cot\theta}{r}+\frac{VW}{r}\right)=-\frac{\partial p}{r\partial\theta} \qquad (1)$$

for meridional flow, neglecting terms involving derivatives with respect to longitude since these become identically zero on averaging, later in the analysis, around latitude circles. This averaging masks out the ocean to continent topographical and thermal effects but comparison of hemispheres suggests that these do not play a fundamental part in forming the main turbulent structure of the atmosphere. We also have

$$\rho\left(\frac{\partial U}{\partial t}+W\frac{\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial \theta}+\frac{UW}{r}+\frac{UV}{r}\cot\theta\right)=0 \qquad (2)$$

for zonal flow,

$$\rho\left(\frac{\partial W}{\partial t}+W\frac{\partial W}{\partial r}+\frac{V}{r}\frac{\partial W}{\partial \theta}-\frac{U^2+V^2}{r}\right)=-g\rho-\frac{\partial p}{\partial r} \qquad (3)$$

for vertical flow, and the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 W)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta\, V)=0 \qquad (4)$$

Following the method adopted by Davies and Oakes (1962), the motion is now regarded as a basic solid rotation flow, with the angular velocity Ω of the earth and an isothermal temperature T_0 together with a flow relative to the rotating earth: this latter flow gives the velocities as observed by earthbound stations. We then write for the solid rotation

$$V = V_0 = 0, \quad U = U_0 = \Omega\, r \sin\theta, \quad W = W_0 = 0; \quad \rho = \rho_0, \quad p = p_0,$$

both these being functions of position, so that

$$-\, r\, \Omega^2 \sin^2\theta = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial r} - g\,, \qquad (5)$$

and

$$-\, r\, \Omega^2 \sin\theta\cos\theta = -\frac{1}{\rho_0}\frac{\partial p_0}{r\partial\theta}\,. \qquad (6)$$

Using $p_0 = \rho_0 R T_0$, where T_0 is a constant, we note from Eq. (5) that p_0 and ρ_0 are both effectively exponentially decreasing functions in the vertical direction. Writing $V = v$, $U = \Omega r \sin\theta + u$, $W = w$, $p = p_0 + p_*$, $\rho = \rho_0 + \rho_*$, the equations of flow relative to the rotating earth are conveniently written in the form

$$\rho\, I_1 = \rho_*\, r\, \Omega^2 \sin\theta\cos\theta - \frac{\partial p_*}{r\partial\theta}\,, \qquad (7)$$

for meridional flow,

$$\rho\, I_2 = \rho_*\,(r\,\Omega^2\sin^2\theta - g) - \frac{\partial p_*}{\partial r}\,, \qquad (8)$$

for vertical flow and

$$\frac{Du}{Dt} + \frac{uw}{r} + \frac{uv}{r}\cot\theta + 2\,\Omega\, w\sin\theta + 2\,\Omega\, v\cos\theta = 0 \qquad (9)$$

for zonal flow, where

$$I_1 \equiv \frac{Dv}{Dt} - \frac{u^2\cot\theta}{r} + \frac{vw}{r} - 2\,\Omega\, u\cos\theta,$$

and

$$I_2 \equiv \frac{Dw}{Dt} - \frac{(u^2+v^2)}{r} - 2\,\Omega\, u\sin\theta.$$

Eliminating pressure at this stage between Eqs. (7) and (8), we obtain

$$\frac{\partial}{\partial r}(\rho r I_1) - \frac{\partial}{\partial\theta}(\rho I_2) = r^2\,\Omega^2\sin\theta\cos\theta\,\frac{\partial\rho_*}{\partial r} + (g - \Omega^2\, r\sin^2\theta)\,\frac{\partial\rho_*}{\partial\theta}\,, \qquad (10)$$

which after some algebra is written

$$\left[\left(1+\frac{\rho_*}{\rho_0}\right)\left(\frac{\partial}{\partial r}+\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial r}\right)+\frac{\partial}{\partial r}\left(\frac{\rho_*}{\rho_0}\right)\right](rI_1) - \left[\left(1+\frac{\rho_*}{\rho_0}\right)\left(\frac{\partial}{\partial\theta}+\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\frac{\rho_*}{\rho_0}\right)\right](I_2)$$
$$= (g - \Omega^2\, r\sin^2\theta)\,\frac{\partial}{\partial\theta}\left(\frac{\rho_*}{\rho_0}\right) + (\Omega^2\, r^2\sin\theta\cos\theta)\,\frac{\partial}{\partial r}\left(\frac{\rho_*}{\rho_0}\right) \qquad (11)$$

In order to simplify the equation we obtain a relationship between ρ_* and the other variables. Assuming hydrostatic equilibrium, the gas law $p = \rho RT$, and taking $r = a + z$, we first obtain

$$p = p_h \exp\left(-\frac{g}{R}\int_h^z \frac{dr}{T}\right) \qquad (12)$$

where h is the upper boundary (assumed independent of θ) of our atmospheric layer, with an associated pressure p_h. In order to obtain a computationally convenient form for the required relation between ρ_* and T we identify p_h with p_0 along $z = h$. This implies that our convectively active model troposphere is not influenced by pressure variations in the uniform rotational flow above the model tropopause and leads to a slow variation of p_h with θ. The effect of the approximation on the main characteristics of the solution is likely to be small compared with that of the primary parameters of flow, in which our main interest lies. We then obtain

$$1 + \frac{\rho_*}{\rho_0} = \frac{T_0}{T} \exp\left\{\frac{g}{RT_0}\int_h^z \left(1 - \frac{T_0}{T}\right) dz\right\}, \quad (13)$$

with the use of which ρ_*/ρ_0 may be eliminated from the equations of motion. Using Eq. (13) we find that the maximum value of the magnitude of ρ_*/ρ_0 is about $\frac{1}{8}$, that $(1/\rho_0)(\partial\rho_0/\partial r) = -gRT_0 \sim 1/h$, that $\partial/\partial r\,(\rho_*/\rho_0)$ is about $\frac{1}{8}$ of $1/h$, that $\partial/\partial\theta\,(\rho_*/\rho_0)$ is about $\frac{3}{10}$, and neglecting $r\,\Omega^2 \sin^2\theta$ by comparison with g, Eq. (11) may be reduced to

$$\left(\frac{\partial}{\partial r} - \frac{g}{RT_0}\right)(rI_1) - \left(\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial\theta} + \frac{\partial}{\partial\theta}\right)I_2 = g\frac{\partial}{\partial\theta}\left(\frac{\rho_*}{\rho_0}\right) + r^2\,\Omega^2 \sin\theta\cos\theta\,\frac{\partial}{\partial r}\left(\frac{\rho_*}{\rho_0}\right). \quad (14)$$

We now substitute into Eqs. (4), (9) and (14), $u = \bar{u} + u'$, $v = \bar{v} + v'$, $w = \bar{w} + w'$, where $\bar{u}$, $\bar{v}$, $\bar{w}$ denote mean values over time periods long enough for $\overline{u'} = \overline{v'} = \overline{w'} = 0$; it is assumed that these mean values are independent of successive averaging periods and that eddy fluctuations in ρ do not influence significantly the ensuing Reynolds stresses. These stresses form a complex system involving many terms of the type appearing in Eq. (11) of Davies and Oakes (1962). Although it is certain that the presence of density within the correlations is of importance, particularly in regions of mixing, lack of knowledge makes it necessary to remove the ρ factor from the eddy terms. In order to reduce the system of mean motion equations to a mathematically tractable form and to extract the most significant terms three basic approximations are made :

(i) Since the ratio (tropospheric thickness)/(radius of the earth) is very small, some of the usual boundary layer approximations are made and consequently vertical gradients of velocity are retained but meridional gradients of velocity are neglected. The three shearing types of Reynolds stress $\overline{\rho\, u'\, v'}$, $\overline{\rho\, u'\, w'}$, $\overline{\rho\, v'\, w'}$ are then assumed to be dominant, and the terms involving ρ' and normal stresses $\overline{u'\, u'}$, $\overline{v'\, v'}$, $\overline{w'\, w'}$ are neglected; the bar denotes a long time period average;

(ii) ρ_*/ρ_0, a function of r and θ, is neglected by comparison with unity in the inertial terms;

(iii) the height of the tropopause is assumed to be independent of θ; this approximation is unlikely to alter significantly the main characteristics of the mean flow solution, which is primarily controlled by Ω, ΔT, and the turbulence parameters.

Writing $r = a + h\zeta$ and using these approximations together with the fact that $\bar{w}$ is much smaller than $\bar{u}$ or $\bar{v}$, Eq. (14) becomes

$$\left(\frac{\partial}{\partial\zeta} - \frac{gh}{RT_0}\right)\left(\frac{\partial}{\partial\zeta}\overline{w'\, v'} - \bar{u}^2\frac{h}{a}\cot\theta - 2\Omega\,\bar{u}\,h\cos\theta\right)$$
$$= g\frac{h^2}{a}\frac{\partial}{\partial\theta}\left(\frac{\rho_*}{\rho_0}\right) + ah\,\Omega^2\sin\theta\cos\theta\,\frac{\partial}{\partial\zeta}\left(\frac{\rho_*}{\rho_0}\right), \quad (15)$$

$(h/a)\,\bar{u}^2\cot\theta$ being retained as the dominant non-linear inertia term (apart from the non-linear Reynolds stress terms). A detailed analysis of the relative magnitudes of the various terms is given by Williams (1964). Eq. (9) for the zonal motion becomes

$$\frac{1}{h}\frac{\partial}{\partial\zeta}(\overline{u'\, w'}) = -\left(\frac{1}{a}\frac{\partial}{\partial\theta}\overline{u'\, v'} + \frac{2\cot\theta}{a}\overline{u'\, v'} + 2\Omega\,\bar{u}\cos\theta\right). \quad (16)$$

In order to close the dynamical equations mathematically we now relate the three eddy shearing stresses to suitable 'driving' parameters. We write

$$\overline{u'w'} = -K_{ZV}\frac{\partial u}{\partial r}, \qquad (17)$$

$$\overline{v'w'} = -K_{MV}\frac{\partial v}{\partial r}, \qquad (18)$$

and

$$\overline{u'v'} = -K_{ZM}(\zeta)\,\Omega\frac{\partial \overline{T}}{r\partial\theta}, \qquad (19)$$

where K_{ZV} and K_{MV} are eddy viscosity coefficients, of the usual form, associated with vertical transfer of zonal momentum and of meridional momentum respectively; this means that in the model considered vertical flow is controlled by mechanical or 'frictional' forces, but the meridional flow, described by $\overline{u'v'}$, is controlled by thermal, i.e. baroclinic, forces. In this way we distinguish, following Saltzman (1957), between two basic regimes of atmospheric turbulence. On the one hand 'direct' turbulence is produced by direct conversion of other forms of energy, such as available potential energy from solar and terrestrial heating, to turbulent energy; on the other hand indirect turbulence is formed as a result of kinetic energy transfer from larger scales of turbulence motion to smaller scales. An example of direct turbulence is the cyclonic-scale disturbances in mid-latitudes which grow at the expense of potential energy due to baroclinic instability, whereas the energy of the frictionally induced micro-turbulence near the ground surface is fed from the energy of the large-scale global motions and is an example of indirect turbulence. Most of the classical treatments of turbulence have been concerned with the latter case and are not generally applicable to the direct type of turbulence.

The dependence assumed in Eq. (19) is based on the well-known notion (see e.g., Thompson 1961) that the energy of growth and decay of the large scale eddies of the circulation (cyclones, depressions, etc.) is strongly dependent on the fundamental parameter of meridional temperature gradient, this being a measure of the mean available potential energy (see e.g., Fjørtoft 1951). In fact the distribution of $\overline{u'v'}$ reproduced by the model through the meridional temperature gradient does closely resemble that produced by observational analyses. If, on the other hand, $\overline{u'v'}$ behaved as a 'friction' stress of indirect turbulence, it would have the effect of retarding the zones of most rapid rotation and of accelerating the less rapidly rotating ones, so as to cause the whole atmosphere to adopt a more nearly uniform angular velocity, i.e., lateral friction would cause the flow of angular momentum northward and also southward from the zone of most rapid atmospheric rotation. However, there exists a strong northward eddy transport of angular momentum from low latitudes, i.e. up the gradient of mean velocity; this is clearly thermally activated and falls into the 'direct' type of turbulence which we have described by Eq. (19). Further justification of this basic postulate can be derived from the seasonal variations of eddy stress, as these follow the seasonal variations of temperature gradient: in winter when greater *thermal* activity prevails, both gradient and stress are twice their summer values. The latitudinal distributions of $\overline{u'v'}$ and temperature gradient are closely similar (see Starr 1953).

Approximate numerical values of K_{ZV} and $K_{ZM}(\zeta)$, the latter being the parameter associated with meridional flow of zonal velocity, can be obtained from the analysis of Tucker (1960). We obtain K_{ZV} and $K_{ZM}(\zeta)$ to be about 10^6 and 10^{17} c.g.s. units, respectively. An order of magnitude value of K_{MV} of about 10^8 can be derived from an analysis of $\overline{v'w'}$ by Molla and Loisel (1962), although we note that $\overline{v'w'}$ is difficult to compute because of the uncertainty and small magnitudes of v and w, so that Eq. (18) is taken only as an initial approximation. In a more advanced and more realistic mean flow model these important parameters should be taken to be functions of the thermal stability and latitude. It is also likely that Eqs. (17), (18) and (19) should be subdivided to cover different sections of the spectra. Observational analyses show that $\overline{u'v'}$ increases strongly with height; this has been taken account of in the present numerical calculations by expressing $K_{ZM}(\zeta)$ as a parabolic function of height. The physical situation is that the large scale global eddies (the cyclonic systems) are created near the surface by large meridional temperature contrasts and, as the thermal winds increase with height, it is possible that the eddy velocity components also increase until they tend to begin to be damped out again in the region of the tropopause; consequently eddy kinetic energy is more easily produced at higher levels, and $\overline{u'v'}$ will attain larger values.

The positive values assumed for K_{ZV}, K_{MV}, and $K_{ZM}(\zeta)$ lead to the actual sign distribution (i) of $\overline{u'v'}$ over the whole atmosphere (ii) of $\overline{u'w'}$ over most of region, except in areas where strong thermal effects could invalidate Eq. (17), (iii) of $\overline{v'w'}$ in the region of strongest vertical gradient in the meridional velocity – at some latitudes the dependence on velocity gradient is weaker and is probably completely masked by thermal forces.

It is convenient at this stage to express the dynamical Eqs. (15) and (16) in terms of the non-dimensional variables $u = (K_{MV}/gh^2)\,\overline{u}$, $v = (K_{MV}/gh^2)\,\overline{v}$, $w = (K_{MV}/gh^2)\,\overline{w}$, and $P = \Omega h^2/K_{MV}$; so that we obtain our basic equations in the forms

$$\left(\frac{\partial}{\partial\zeta} - \frac{gh}{RT_0}\right)\left(\frac{\partial^2 v}{\partial\zeta^2} + \frac{gh^4}{aK^2_{MV}}u^2\cot\theta + 2\,Pu\cos\theta\right) = -\frac{h}{a}\frac{\partial}{\partial\theta}\left(\frac{\overline{\rho_*}}{\rho_0}\right) - \left(\frac{aK^2_{MV}}{gh^4}\right)P^2\sin\theta\cos\theta\frac{\partial}{\partial\zeta}\left(\frac{\overline{\rho_*}}{\rho_0}\right) \quad (20)$$

and

$$\frac{\partial^2 u}{\partial\zeta^2} = 2\frac{K_{MV}}{K_{ZV}}P\,v\cos\theta - \frac{K_{ZM}(\zeta)}{g\,a^2\,h^2}\frac{K^2_{MV}}{K_{ZV}}P\left(\frac{\partial^2\overline{T}}{\partial\theta^2} + 2\cot\theta\frac{\partial\overline{T}}{\partial\theta}\right) \quad (21)$$

3. The formulation of a thermodynamic equation and parameterization of the atmospheric heating function

The first law of thermodynamics can be written in the form

$$\frac{dq}{dt} = c_p\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt}, \quad (22)$$

where dq/dt is the rate at which thermal energy is added to a unit mass of air by radiation, molecular conduction and viscous dissipation. Using the continuity equation this can be expressed in the form

$$\frac{\partial}{\partial t}(\rho q) + \nabla(\rho q\mathbf{V}) = c_p\left[\frac{\partial}{\partial t}(\rho T) + \nabla(\rho T\mathbf{V})\right] - \frac{\partial p}{\partial t} - \mathbf{V}\cdot\nabla p \quad (23)$$

where $\mathbf{V}$ is the total velocity. Applying the Reynolds averaging technique to Eq. (23) (see e.g., Van Mieghem 1958) and simplifying, using the hydrostatic assumption and re-arranging in polar co-ordinates, we obtain

$$\frac{1}{\overline{\rho}\,r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\overline{\rho v}\,\overline{T} + \overline{(\rho v)'T'}\right)\right] = \frac{1}{c_p}\left[-\frac{c_p}{\overline{\rho}r^2}\frac{\partial}{\partial r}\left\{r^2\left(\overline{\rho w}\,\overline{T} + \overline{(\rho w)'T'}\right)\right\} + \frac{\partial\overline{q}}{\partial t} + \overline{\mathbf{V}}\cdot\nabla\overline{q} + \frac{1}{\overline{\rho}}\nabla\overline{(\rho q)'\mathbf{V}'} - g\overline{w} + \overline{w'\frac{\partial p'}{\partial r}}\right], \quad (24)$$

where the left-hand side terms represent meridional transfer by the large-scale eddies and mean motion and the right-hand side terms represent the heating, due to adiabatic effects, radiation exchanges, oceanic transports, sensible heat transport by large-scale eddy transport, latent heat condensation effects and small-scale heat diffusion.

In order to formulate a thermodynamic equation, which is mathematically consistent with Eqs. (20) and (21), in the sense that terms involving $\overline{w}$ have been removed and to simplify a difficult mathematical problem, we also absorb the influence of vertical velocity in an empirical expression adopted for the right-hand side of Eq. (24). We compensate, to some extent, for the omission of vertical transport mechanisms by specifying the vertical distribution of the empirical heating function $Q(\theta, \zeta)$ on the right-hand side of Eq. (24), which we then write, using non-dimensional velocities, in the form

$$\frac{1}{\overline{\rho}\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\overline{\rho v}\,\overline{T}+\overline{(\rho v)'\,T'}\right)\right]=\left(\frac{K_{MV}}{gh^2}\right)a\,Q(\theta,\zeta) \qquad (25)$$

The limited problem remaining is still meteorologically interesting as it contains the main forcing (and probably the most interesting) terms of the circulation, i.e., $\overline{u'\,v'}$ and $\overline{v'\,T'}$.

The cyclonic eddies producing the stress, described by $\overline{u'\,v'}$ are primarily responsible for the meridional eddy transfer of heat and to close the thermodynamical equation mathematically, we assert that the main driving parameter influencing heat flow by turbulent eddies is the meridional temperature gradient (i.e. a diffusion mechanism) and we write, firstly

$\overline{vT}=\overline{v}\,\overline{T}+\overline{v'\,T'}$, where, in terms of non-dimensional velocities,

$$\overline{v'\,T'}=-\left(\frac{K_{MV}}{gh^2}\right)K_{MT}\frac{\partial\overline{T}}{a\partial\theta} \qquad (26)$$

The computations of Adem (1962) and others have confirmed observations that the main mechanism of meridional transport in the troposphere is down-gradient turbulent diffusion of heat through the large eddies: using a form similar to Eq. (26) and a numerical value of about $2{\cdot}5 \times 10^{10}$ for K_{MT}, Adem successfully reproduced some atmospheric mean temperature profiles. Although most of the meridional momentum transfer takes place in the upper troposphere (due to increasing thermal wind effects with height), the heat transfer is more uniformly distributed over the depth of the troposphere, and so K_{MT} is taken in Eq. (26) to be independent of height.

In dealing with the total heating function, Q, it has been taken firstly to be a prescribed function of height and latitude and incorporates estimates of all the factors contributing to heating, as specified by, e.g., Smagorinsky (1963) in his account of a time-dependent general circulation model containing the main characteristic factors involved, viz., short wave and long wave radiation effects, latent and sensible heat flux and condensation effects. Given Q as a geographically determined total heating function (of the type used by Phillips, 1956) we then obtain a series solution for u, v and T of Eqs. (20), (21) and (25) and test the effects of varying some of the parameters involved. It has been shown, Smagorinsky (1963), that the atmospheric heating rate can be expressed in terms of the radiation excess of atmosphere and earth and a factor depending on the difference between condensation and precipitation. Although the value of the radiation term is established, the contribution of the condensation effects is only roughly known. For comparison purposes, the effects on the velocity and temperature of using different estimates of the heating rate are studied. To obtain a more incisive parameterization of the atmospheric heating, a second heating function in which sea-air interchange is made a function of the surface velocity and temperature, is used. Thus we split Q into $Q_{S+C}(\overline{\mathbf{V}}, T)$, the heating due to sensible heat transport from the ocean and to condensation of water vapour and $Q_R(r, \theta)$ due to the other heating components.

The empirical expression

$$[Q_S]_0 = c_1\,|\mathbf{V}_s|\,(T_g - T_s)\ , \qquad (27)$$

has been found to be fairly reliable for climatological computations of sensible heat transfer from water surfaces by turbulent exchange: $\overline{T}_g$ and T_s denote mean earth boundary temperature and mean air surface temperature respectively and $\mathbf{V}_s$ the mean surface air velocity. The exchange over land is, of course, more complicated and we assume that our model is entirely oceanic. A similar expression for latent heat surface to air transfer is

$$[Q_L]_0 = c_2 |\bar{\mathbf{V}}_s| (E - e) , \qquad (28)$$

where E and e denote saturated vapour pressure at the water surface temperature and vapour pressure at the air surface temperature respectively. These empirical formulae are discussed by a number of authors (e.g., Jacobs 1951).

The latent heat associated with water vapour transferred from the surface is released at the time of condensation in the form of cloud and it is only the component falling out as precipitation which contributes permanently to the atmospheric heating: a cloud which forms and subsequently dissipates has contributed zero energy because the same amount of energy is used to evaporate the cloud as was released through its formation. Consequently only the component of the transferred water vapour, which falls to the surface as precipitation, should strictly be considered. We note also that latent heat inserted at one latitude is released as precipitation at another latitude. These are complex mechanisms and present us with a very difficult problem of mathematical formulation, particularly over the climatological scale, which we have not yet solved. We note, however, that Smagorinsky (1963) achieves an approximate formulation of the condensation problem by relating condensation heating to a function of the vertical gradient of potential temperature. The statistical effects of condensation processes in the climatological problem are more difficult to formulate. However, in order to probe a little into the sea-air interaction these expressions are linked together by means of an approximation based on the Magnus equation, that $E - e$ is proportional to $(\bar{T}_g - \bar{T}_s)$. We combine Eqs. (27) and (28) into one expression which we take as a measure of sea-air interchange, namely

$$[Q_{S+L}]_0 = c_3 |\bar{\mathbf{V}}_s| (\bar{T}_g - \bar{T}_s) , \qquad (29)$$

where the proportionality factor c_3 for the actual atmosphere can be estimated from climatological observations and a suitable Q_{S+C} function can be obtained by multiplying $(Q_{S+L})_0$ by a function $J^* (\zeta)$ prescribed empirically. We note that the modulus of $\bar{\mathbf{V}}_s$ introduces mathematical difficulties for the expansion type of solution: $c_3 |\bar{\mathbf{V}}_s|$ is replaced by $(c_4 + c_5 \bar{\mathbf{V}}_s^2)$ and values of c_4 and c_5 are chosen to obtain a good fit over the observed $\bar{\mathbf{V}}_s$ range. The problem is further complicated by the necessity of maintaining zero net average heating over the whole atmosphere. With present techniques, computer capability was insufficient to deal with the requirements of this balance constraint for this second heating form. However, in order that some interaction could be studied the feedback component, Eq. (29), was taken to be small by reducing the value of c_3 to a small magnitude.

4. The boundary conditions

Two dynamical conditions are used:

(i) $$\frac{\partial u}{\partial \zeta} = \frac{\partial v}{\partial \zeta} = 0, \quad \text{on} \quad \zeta = 1{\cdot}0, \qquad (30)$$

representing the assumption of zero vertical mixing at the tropopause.

(ii) The boundary condition at the earth's surface, $\zeta = 0$, is important, since this controls the flow of angular momentum between the earth and atmosphere. In the free atmosphere friction due to viscosity is generally unimportant and is omitted from the equations but friction near the surface is responsible for dissipation of kinetic energy brought to the surface by the eddies. Since this frictional surface boundary layer is thin, we neglect its thickness by comparison with the tropospheric thickness and introduce its effects through a frictional surface boundary condition. In this surface layer the numerical values of the various turbulence parameters will be different from those found appropriate in the free atmosphere; they are denoted by $(K_{ZV})_0$ and $(K_{MV})_0$ and are both given the value $10^3\,\text{cm}^2\,\text{sec}^{-1}$, which is approximately the value obtained for eddy viscosity just above the sea (Taylor 1916).

At $\zeta = 0$, we then assume the Reynolds stresses depend linearly on the appropriate components of velocity, i.e.,

$$(K_{ZV})_0 \frac{\partial u}{\partial \zeta} = 2ku, \qquad (K_{MV})_0 \frac{\partial v}{\partial \zeta} = kv \quad . \quad . \quad . \quad (31)$$

where k is estimated, over the known range of surface velocities, from the more accurate empirical formula in which the stresses are known to be proportional to the squares of velocities; the 2-factor in the zonal component compensates for the higher zonal velocities. This linearization simplifies the solutions and seems unimportant in view of Smagorinsky's (1963) result that the surface torque remains more or less constant, irrespective of the form chosen to represent it.

Since there is no net flow of mass over a period of a year (in our stationary model) across a latitude circle, we require the continuity condition that

$$\int_0^1 \rho v \, d\zeta = \int_{p_s}^{p_h} v \, dp = 0 \quad . \quad . \quad . \quad . \quad (32)$$

where p_s is the surface pressure: this becomes on approximating $\rho = \rho_0$, $\int_0^1 v\, e^{-b\zeta}\, d\zeta = 0$, where $b = gh/R\,T_0$ measures the exponential decrease of ρ with ζ.

The flux divergence of heat in the oceans is a fundamental factor in influencing the interaction, but in this limited problem we assume the mean ocean temperature is prescribed in the empirical form

$$T_g/T_0 = 1 + (\Delta T/T_0)\, F_g(\theta) \quad . \quad . \quad . \quad . \quad (33)$$

since in the case $\Delta T = 0$ the sea temperature has a constant value. $F_g(\theta)$ matches the observed mean sea temperatures. We take $F_g(\theta) = \frac{1}{3} - \frac{1}{6}\cos^2\theta - \frac{4}{6}\cos^4\theta$ as a suitable approximation. We find, however, that a further specification of boundary temperature is required to define the solution completely. The form of the thermodynamic Eq. (25) has no mechanism to define the vertical distribution of temperature and is only capable of giving the meridional gradient of temperature. It is necessary, therefore, to prescribe the absolute temperature distribution along the vertical at the Equator, $\theta = 90°$, $\zeta = 0$ to $1{\cdot}0$. The model is then able to specify temperature at all other points. When the exchange formula Eq. (29) is used in the heating formulation, there is an amount of vertical resolution and it is only necessary to prescribe the vertical temperature gradient at $\theta = 90°$.

5. Solution of the dynamical and thermodynamical equations

Eqs. (13), (20), (21), (26) with (27), form a consistent set whose solution is required. Crucial basic parameters influencing the general circulation are clearly the earth's angular velocity, Ω, and ΔT, the mean meridional temperature difference between equator and pole, which must control the mean degree of large-scale turbulence. The results of Davies and Oakes (1962) suggested that a solution involving an expansion in powers of $P = \Omega\, h^2/K_{MV}$ might be sufficiently convergent. In addition it is noted that, although ΔT for summer and winter differ by a factor of 2, the main characteristics of the circulation do not differ from winter to summer as much as the difference in ΔT might suggest. This indicates that a double expansion solution in P and ΔT might be sufficiently convergent over realistic values of both P and ΔT to calculate theoretical distributions of u, v and T.

We therefore write

$$\overline{T}/T_0 = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^i P^j \, T_{ij}(\theta, \zeta) \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Substituting from Eq. (34) into Eq. (13) we obtain

$$\frac{\overline{p}_*}{\rho_0} = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^i P^j \left(- T_{ij} + b \int_1^{\zeta} T_{ij} \, d\zeta + S_{ij}\right) \quad , \quad . \quad . \quad (35)$$

where the S_{ij} are remainder functions dependent upon lower coefficients of T_{ij}, e.g.

$$S_{1j} \equiv 0 \text{ for all } j,$$

$$S_{20} = T_{10}^2 - b \int_1^{\zeta} T_{10}^2 \, d\zeta - b \, T_{10} \int_1^{\zeta} T_{10} \, d\zeta + \frac{b^2}{2} \left(\int_1^{\zeta} T_{10} \, d\zeta\right)^2, \quad . \quad (36)$$

and are easily obtained by expanding Eq. (13). For the velocity variables we take

$$v = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^i P^j \, V_{ij}(\theta, \zeta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$u = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^i P^j \, U_{ij}(\theta, \zeta). \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

With the uniform conditions assumed in this model, a degree of symmetry is present with respect to Ω. Thus if Ω becomes $-\Omega$ we would expect v, T, $\overline{p}$ to be unaltered but u to reverse sign so that the odd powers of P vanish from the v, T, $\overline{p}$ expansions and the even ones from the u expansions. This is verified by the equations.

To complete the expansions we consider the heating function in the form

$$Q = Q_R(\theta, \zeta) + (c_4 + c_5 \, \mathbf{V}_s^2)(\overline{T}_g - \overline{T}_s) \, J^*(\zeta). \quad . \quad . \quad . \quad (39)$$

To expand this we note that Q_R must be dependent upon ΔT so, assuming a linear dependence, we replace it by $(\Delta T/T_0) \, Q^*_R$. Also we note that as $(c_4 + c_5 \, \mathbf{V}_s^2)$ has replaced $c_3 |\overline{\mathbf{V}}_s|$, c_4 and c_5 must incorporate powers of $(\Delta T/T_0)$ so that the lowest term in $(c_4 + c_5 \, \overline{\mathbf{V}}_s^2)$ depends on $(\Delta T/T_0)^1$.

Thus we write

$$Q = \left(\frac{\Delta T}{T_0}\right) Q^*_R(\theta, \zeta) + \left(\frac{\Delta T}{T_0}\right)^2 [Q_{S+L}]_0 \, J^*(\zeta) \quad . \quad . \quad . \quad (40)$$

where

$$[Q_{S+L}]_0 = \left[c_7 + c_8 \left(\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^{i-1} P^{2j} \, V_{i(2j)}\right)^2 + c_8 \left(\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^{i-1} P^{2j+1} \, U_{i(2j+1)}\right)^2\right]$$

$$\times \left[F_g - \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{\Delta T}{T_0}\right)^{i-1} P^{2j} \, T_{i(2j)}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

c_7, c_8 being related to c_4 and c_5 respectively. Then Eq. (25) may be expressed in the form

$$v \frac{T}{T_0} - \frac{f}{c} \frac{\partial}{\partial \theta} \left(\frac{\overline{T}}{T_0}\right) = \frac{M}{\sin \theta} \int_{\pi/2}^{\theta} \sin \theta \left[\left(\frac{\Delta T}{T_0}\right) Q^*_R + \left(\frac{\Delta T}{T_0}\right)^2 Q^*_{S+L}\right] d\theta \quad , \quad . \quad (42)$$

with $\quad Q^*_{S+L} = [Q_{S+L}]_0 \, J^*(\zeta) \, (\Delta T/T_0)^{-2}$,

where $f = \dfrac{h}{a}$, $M = \dfrac{a K_{MV}}{g h^2 \, T_0}$, and $c = \dfrac{g h^3}{K_{MT} \, K_{MV}}$ is a Rayleigh type number.

Substituting the series forms for u, v, T into Eqs. (20), (21) and (42) and comparing like powered coefficients leads to a system of *linear* equations for the coefficients. Introducing the operators

$$D \equiv \frac{\partial}{\partial \zeta} \quad \text{and} \quad \Phi(D)\, V \equiv (D-b)\, D^2 V - c\left(V - b\int_1^{\zeta} V\, d\zeta\right),$$

we obtain the following equations for the coefficients. Firstly in the $(\Delta T/T_0)^1$ series, for the P^0 term we have

$$\Phi(D)\, V_{10} = -\frac{Mc}{\sin\theta}(D-b)\int_{\pi/2}^{\theta}\int_1^{\zeta} Q^*_R(\theta,\zeta)\sin\theta\, d\theta\, d\zeta \quad . \quad . \quad . \quad (43)$$

from which V_{10} can be evaluated, and

$$\frac{\partial T_{10}}{\partial \theta} = \frac{c}{f} W_{10}(\theta,\zeta), \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where

$$W_{10}(\theta,\zeta) = V_{10} - \frac{M}{\sin\theta}\int_{\pi/2}^{\theta} Q^*_R \sin\theta\, d\theta \quad . \quad . \quad . \quad (45)$$

is known; hence T_{10} may be evaluated on defining the absolute temperature values at the Equator with $Q_{S+L} \equiv 0$, or on imposing a zero net global heating constraint with $Q_{S+L} \neq 0$. For the P^1 term we have

$$D^2 U_{11} = \frac{K_{MV}}{K_{ZV}}\left[2\cos\theta\, V_{10} - \frac{K_{ZM}(\zeta)}{K_{MT}}\frac{T_0}{a}\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\, W_{10})\right], \quad . \quad . \quad (46)$$

and for the P^{2n} term, where $n \geqslant 1$, we have

$$\Phi(D)\, V_{1(2n)} = (D-b)\left[-2\cos\theta\, U_{1(2n-1)} + \frac{aK^2_{MV}}{gh^4}\sin\theta\cos\theta\, T_{1(2n-2)}\right] \quad , \quad (47)$$

and

$$T_{1(2n)} = \frac{c}{f}\int_{\pi/2}^{\theta} V_{1(2n)}\, d\theta \quad . \quad . \quad . \quad . \quad . \quad (48)$$

For P^{2n+1} we have

$$D^2 U_{1(2n+1)} = \frac{K_{MV}}{K_{ZV}}\left[2\cos\theta\, V_{1(2n)} - \frac{K_{ZM}(\zeta)}{K_{MT}}\frac{T_0}{a}\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\, V_{1(2n)})\right] \quad . \quad (49)$$

Proceeding from the basic solution for V_{10} the succeeding terms of the series can thus be evaluated. The terms in the higher $(\Delta T/T_0)$ powers are much more complex in form but can be dealt with by a similar treatment.

The evaluation of the various series coefficients involves the solution of differential equations of the form

$$\Phi(D)\, V_{ij} = R_1(\zeta), \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$T_{ij} = \frac{c}{f}\int_{\pi/2}^{\theta} R_2(\theta)\, d\theta, \quad . \quad . \quad . \quad . \quad (51)$$

$$D^2 U_{ij+1} = R_3(\zeta), \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where $R_{1,2,3}$ can be considered as known forcing functions, given here in the case $i = 1$ by Eqs. (47), (48), (49) and by Williams (1964) for all i, j; the equations are evaluated for all latitudes and heights. The method is repeated for all the i, j powers required to provide convergence i.e. $i = 1, 2, 3, 4$, $j = 0, 2, \ldots, 10$.

To solve Eq. (50), $R_1(\zeta)$ is initially evaluated from its constituents for the 11 points $\zeta = 0{\cdot}0, 0{\cdot}1, \ldots 1{\cdot}0$. (The process is repeated at 5° intervals in the latitudinal variable thus effectively forming a 15×11 grid). Written in full form the equation to be solved is

$$(D - b)\, D^2 V_{ij} - c\,(V_{ij} - b \int_1^{\zeta} V_{ij}\, d\zeta) = R_1(\zeta) \quad , \qquad (53)$$

subject to the boundary conditions in Eqs. (30), (31) and (32) with $R_1(\zeta)$ as a known forcing function. Introducing the parameter $\gamma^3 = c$ and differentiating, Eq. (53) has a solution in the form

$$V_{ij} = A_1 e^{b\zeta} + A_2 e^{\gamma\zeta} + e^{-(\gamma\zeta/2)} \left[A_3 \cos\left(\frac{\sqrt{3}}{2}\gamma\zeta\right) + A_4 \sin\left(\frac{\sqrt{3}}{2}\gamma\zeta\right)\right] + \frac{D R_1}{(D - b)(D^3 - c)} \quad . \qquad (54)$$

An interesting aspect of this equation is noted; if γ is large, the $A_2 e^{\gamma\zeta}$ term will cause an error amplification which suggests that, γ, a type of Rayleigh number depending on $K_{MT} K_{MV}$, is an important parameter. The integration constants are obtained from the boundary conditions. To evaluate Eq. (54) we use the tool of trigonometric interpolation, described by Lanczos (1957) and we represent R_1 by a finite Fourier sine series with linear trend such that R_1 and its representation coincide at the grid points i.e.

$$R_1 = d_1 + d_2\,\zeta + \sum_{s=1}^{9} b_s \sin(s\,\pi\,\zeta) \quad , \qquad (55)$$

where

$$d_1 = R_1(0{\cdot}0), \quad d_2 = R_1(1{\cdot}0) - R_1(0{\cdot}0) \,, \qquad (56)$$

and

$$b_s = \tfrac{1}{5} \sum_{t=1}^{9} \left[R_1(t/10) - d_1 - t\,d_2/10\right] \sin(st\,\pi/10) \qquad (57)$$

With R_1 as the known analytical form in Eq. (55), the solution, Eq. (54), is obtained by analytical operations (reserving numerical operations to evaluation of the constants), in the form

$$V_{ij} = A_1 e^{b\zeta} + A_2 e^{\gamma\zeta} + e^{-(\gamma\zeta/2)} \left[A_3 \cos\left(\frac{\sqrt{3}}{2}\gamma\zeta\right) + A_4 \sin\left(\frac{\sqrt{3}}{2}\gamma\zeta\right)\right] + \frac{d_2}{bc} + \sum_{s=1}^{9} \left[B_s^* \sin(s\pi\zeta) + A_s^* \cos(s\pi\zeta)\right] \,, \qquad (58)$$

where

$$B_s^* = s\pi b_s\, T_s, \qquad A_s^* = s\pi b_s\, S_s,$$

and

$$S_s = \frac{bc + (s\pi)^4}{[b^2 + (s\pi)^2]\,[c^2 + (s\pi)^6]}, \qquad T_s = \frac{s\pi\,[b\,(s\pi)^2 - c]}{[b^2 + (s\pi)^2]\,[c^2 + (s\pi)^6]}.$$

The constants A_1, A_2, A_3, A_4 are obtained by submitting the form in Eq. (58) to the boundary conditions and the undifferentiated Eq. (53). To complete the solution, inversion of the 4×4 matrix associated with the boundary conditions is necessary only once for each c, b, K_s data set.

The general equation for the U_{ij} coefficients of the type of Eq. (49), resembles the basic Eq. (21) if we treat v and T as known and the right-hand side as a forcing function i.e.,

$$\frac{\partial^2 u}{\partial \zeta^2} = R(\theta, \zeta). \qquad (59)$$

Integrating, we obtain

$$\frac{\partial u}{\partial \zeta} = \left[\frac{\partial u}{\partial \zeta}\right]_0 + \int_0^{\zeta} R(\theta, \zeta)\, d\zeta,$$

i.e.,

$$\frac{\partial u}{\partial \zeta} = \frac{2}{K_s} u_s(\theta) + \int_0^{\zeta} R(\theta, \zeta)\, d\zeta, \qquad (60)$$

where $u_s(\theta)$ is the surface zonal velocity, which is given by taking $\zeta = 1{\cdot}0$, and K_s is (the surface value of K_{ZV})/k so that,

$$u_s(\theta) = -\frac{K_s}{2} \int_0^1 R(\theta, \zeta)\, d\zeta\,. \qquad (61)$$

Further integration of Eq. (60) gives

$$u = u_s(\theta)\left(1 + \frac{2\zeta}{K_s}\right) + \int_0^{\zeta}\int_0^{\zeta} R(\theta, \zeta)\, d\zeta\, d\zeta\ , \qquad (62)$$

which shows that individual values of u_{ij} are easily evaluated by numerical integration. From the basic Eq. (21) it is also interesting to note that the surface zonal velocity is given by the expression

$$u_s(\theta) = -K_s\left(\frac{K_{MV}}{K_{ZV}}\right) P \cos\theta \int_0^1 v d\zeta - \frac{K_s K_{MV}}{2ga\, K_{ZV}} \frac{1}{\sin^2\theta} \frac{\partial}{\partial\theta}\left(\sin^2\theta \int_0^1 \overline{u'\, v'}\, d\zeta\right) \quad . \qquad (63)$$

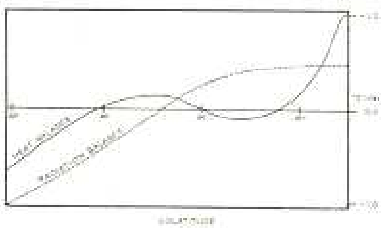


Figure 1. Illustration of the radiative and the net total heating rates in °C day^{-1} (after Smagorinsky, 1963).

6. Results of numerical experiments

Eqs. (13), (20), (21), (25) with (26) form a consistent set whose solution has been obtained by the method discussed in the previous section. Numerical experiments were performed for various parameter values. The most basic parameter of the model is the variation of the heating rate with latitude. Numerical experiments were performed with Q estimated from (a), the radiation balance and (b), the total heat balance. The distributions of these two heating estimates are shown in Fig. 1. In the equatorial zone the gain of heat from the large positive radiation balance is supplemented by a considerable amount from moisture exchange. Northward of this zone is the tropical zone in which there is a constantly diminishing radiational balance with a considerable expenditure of heat used for the moisture exchange. There are no estimates of the vertical variation of the heating rates shown in Fig. 1 so we assume the heating to take the normalized form $J(\zeta)$ in the vertical. In the experiments a linear function $J(\zeta) = 0{\cdot}8 + 0{\cdot}4\,\zeta$ provided the most realistic vertical distribution of the variables. The effect of $J(\zeta)$ is mostly felt in forming the vertical gradient of temperature and it appears that if $\partial J/\partial\zeta$ is not sufficiently large and positive, (i.e. $> 0{\cdot}5$), then the vertical temperature gradient in polar regions reverses in the mid-troposphere.

The fluxes obtained by integrating the rates of Fig. 1 indicate that the total heat balance distribution is roughly half that of the radiational balance. This suggests that the winds necessary to transfer the heat are greatly reduced by the zonally transferring properties of water vapour. For a given meridional temperature gradient the value of K_{MT} for the total heat balance experiments is expected to be half that of radiational balance experiments: in the experiments the values of $2{\cdot}75 \times 10^{10}$ and $4{\cdot}5 \times 10^{10}$ were found to be the most realistic for these respective heating rates in cm sec units.

The lateral stress coefficient $K_{ZM}(\zeta)$ is taken as $K_{ZM}\,\zeta^2$. The parameter K_{ZM} occurs with K_{MT} in the zonal velocity equation and it is noticeable that to reproduce good working models the ratio of K_{ZM}/K_{MT} is approximately constant at $0{\cdot}65 \times 10^7$ for both heating rates, the values of K_{ZM} being $1{\cdot}7 \times 10^{17}$ and $3{\cdot}0 \times 10^{17}$. This indicates that the eddy transfer of momentum is closely linked with the eddy transfer of heat.

The most realistic numerical experimental value of K_{ZV} lies in the range of (5 to 10) $\times 10^5$ in keeping with Tucker's (1960) and (1964) analyses. Experimental results seemed most reasonable with the value of K_{MV} taken as $3{\cdot}5 \times 10^8$. This parameter influences the convergence rate so that its range is limited. Although the magnitudes of the meridional velocity, v, produced with this K_{MV} value are on the small side it is likely that this would be remedied by taking K_{MV} equal to (2 to 3) $\times 10^8$. All values are in cm sec units.

The surface stress parameter K_s (equal to $[K_{MV}]_0/k$) was kept constant at 0·28 for all experiments and gave good surface velocity values. All the parameters are, of course, interrelated through energy partitioning so that the above values are only the most realistic when used together.

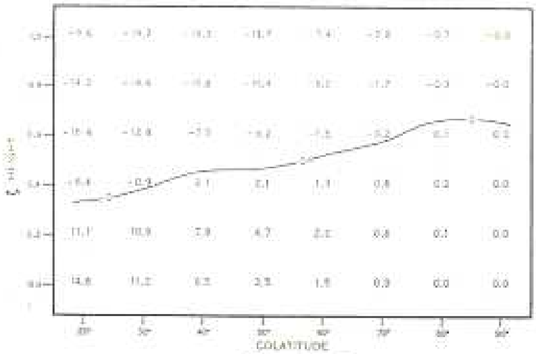
Of the experiments performed, four results are presented here. Experiment 1 is based on the radiative balance. Experiments 2 and 3 were made with the total heat balance estimate and a comparison between their results illustrates parameter effects. Experiment 4 contains a small sea-air exchange feedback. The parameter values are shown in Table 1.

TABLE 1. ILLUSTRATION OF PARAMETER VALUES USED IN CALCULATIONS

Experiment number	K_{MV} cm² sec⁻¹	K_{MT} cm² sec⁻¹	K_Z cm² sec⁻¹	K_{ZM} c.g.s. units	$Q(\theta)$	$J(\zeta)$
1	$3{\cdot}5 . 10^8$	$4{\cdot}5 . 10^{10}$	$1{\cdot}0 . 10^6$	$3{\cdot}0 . 10^{17}$	Radiation	$0{\cdot}8 + 0{\cdot}4\zeta$
2	$3{\cdot}4 . 10^8$	$2{\cdot}6 . 10^{10}$	$0{\cdot}8 . 10^6$	$1{\cdot}7 . 10^{17}$	Heat	1·0
3	$3{\cdot}4 . 10^8$	$2{\cdot}75 . 10^{10}$	$0{\cdot}75 . 10^6$	$1{\cdot}8 . 10^{17}$	Heat	$0{\cdot}8 + 0{\cdot}4\zeta$
4	$3{\cdot}4 . 10^8$	$2{\cdot}75 . 10^{10}$	$0{\cdot}75 . 10^6$	$1{\cdot}8 . 10^{17}$	Feedback	$0{\cdot}8 + 0{\cdot}4\zeta$

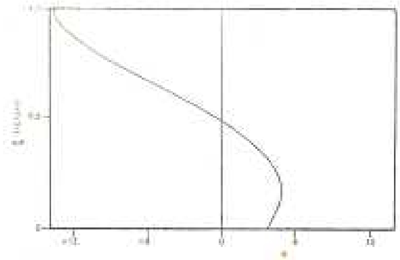
The magnitudes of the coefficients employed in the numerical evaluation were such that convergence with respect to both P and $\Delta T/T_0$ existed to give total variables accurate to within 5%.

In Experiment 1 the radiational heating rate allowed on average a gain of 0·5°C per day at the Equator and a loss of over 1·0°C per day at colatitude 20°. Fig. 2 shows that in response to this heating the mean meridional velocities make up a Hadley cell with a maximum velocity of 19 cm sec⁻¹. The existence of the cell is due to the fact that the poleward eddy heat transfer is insufficient for total balance and must be supplemented by that of the direct Hadley cell. A typical vertical distribution of meridional velocity, Fig. 3, shows that maximum values occur just above the friction layer and at the tropopause. In another experiment with this heating rate the more familiar tricellular pattern was reproduced but as other aspects of the flow were less favourable the results are not presented.



$\bar{V}$ cm sec^{-1}

Figure 2. The meridional wind pattern of Experiment 1. Positive velocities indicate flow towards the Equator. Vertical ordinate is dimensionless height in range surface to tropopause.

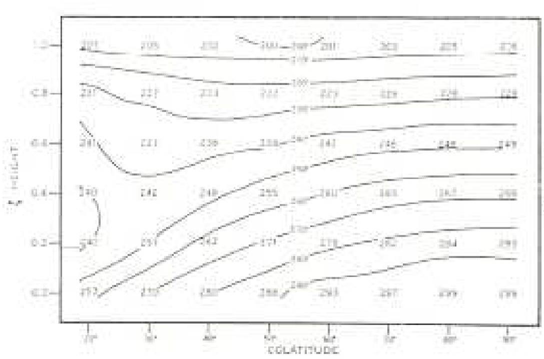


$\bar{V}$ cm sec^{-1}

Figure 3. Illustration of a typical vertical profile of the meridional velocity in the Hadley cell. Vertical ordinate is dimensionless height in range surface to tropopause.

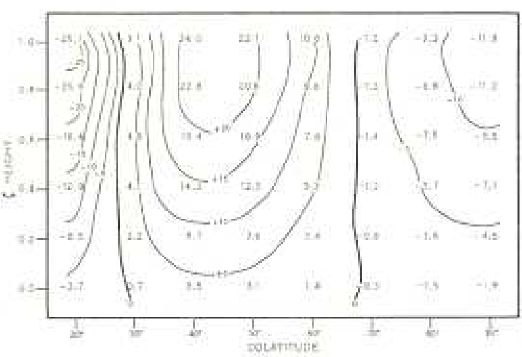
The temperature field produced through the eddy and mean meridional transports of the heat energy from low to high latitudes resembles that of the atmosphere, as shown in Fig. 4. This is encouraging in view of the fact that vertical transfers are only crudely allowed for through the $J(\zeta)$ function. At the surface the temperature varies from 257°K to 299°K and elsewhere the temperature gradients are fairly well behaved with the temperature decreasing poleward in all regions except near $\theta = 20°$ in the upper troposphere. From the meridional gradient of temperature we see that the meridional variation of $\overline{u'v'}$ is in keeping with the observational analyses of Starr (1953), with the sign reversal occuring in the $\theta = 20° - 30°$ interval.

The corresponding zonal wind, Fig. 5, is split into the characteristic east-west-east pattern. In the sub-equatorial regions easterlies are formed up to 12 m sec^{-1} in magnitude. In the $30° - 70°$ colatitude region a wide band of westerlies exists in a distribution closely resembling the atmospheric pattern. The jet-like maximum of 25 m sec^{-1} which appears at $\theta = 45°$ and $\zeta = 1{\cdot}0$ is similar to the winter value for the observed zonal wind. Varying the parameters, particularly K_{ZM}, alters the magnitude and zonal divisioning of this pattern. The equatorial easterlies produced are of the correct magnitude but the distribution is more extensive in the upper layer than in the atmosphere. The position of the maximum



T °K

Figure 4. Temperature field of Experiment 1, in °K; the polar inversion is noteworthy. Vertical ordinate is dimensionless height in range surface to tropopause.



$\bar{U}$ m sec^{-1}

Figure 5. Zonal wind pattern of Experiment 1 in m sec^{-1}, with positive values denoting westerly winds. The jet reaches a maximum of 24 m sec^{-1}. Vertical ordinate is dimensionless height in range surface to tropopause.

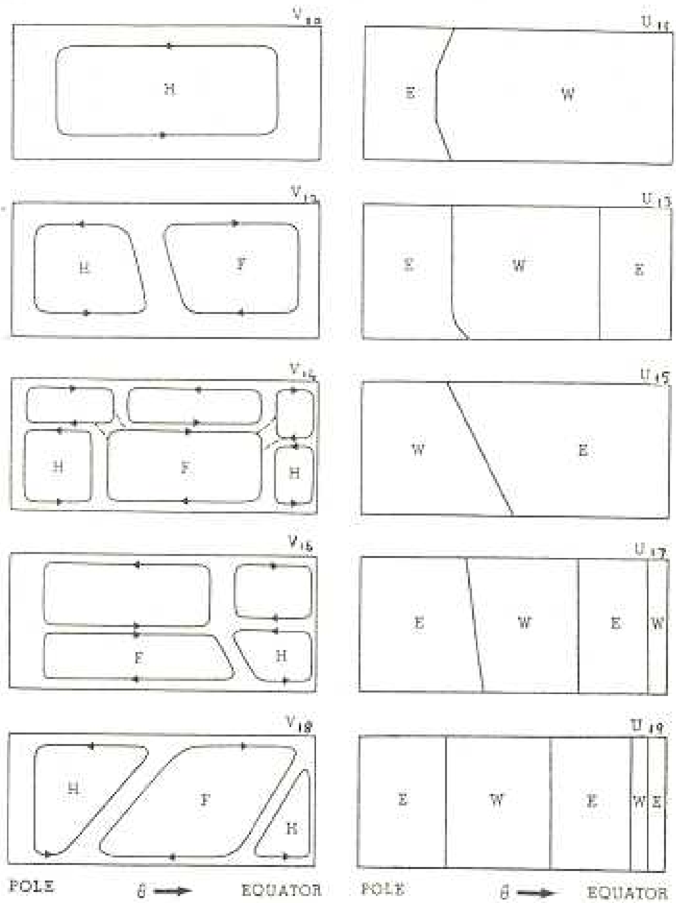


Figure 6. Schematic presentation of some of the series coefficients of v and u. Letters H and F indicate Hadley and Ferrel cells, W and E the westerlies and easterlies respectively. Arrows denote flow direction. Values between 0° and 20° colatitude are extrapolated.

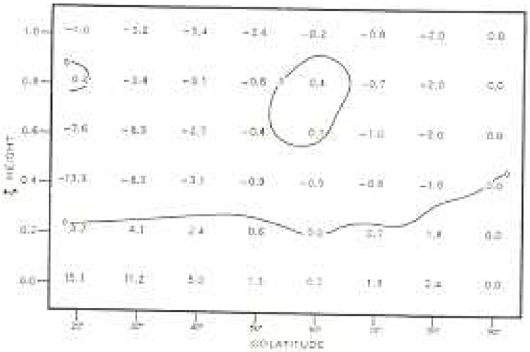
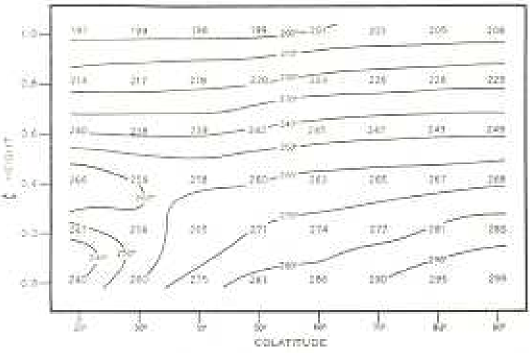
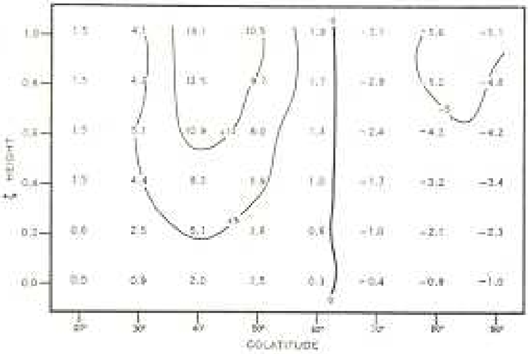


Figure 7. Meridional wind pattern of Experiment 2. Positive values indicate flow towards the Equator. Vertical ordinate is dimensionless height in range surface to tropopause.



T °K

Figure 8. Temperature field of Experiment 2, indicating a polar inversion. Vertical ordinate is dimensionless height in range surface to tropopause.



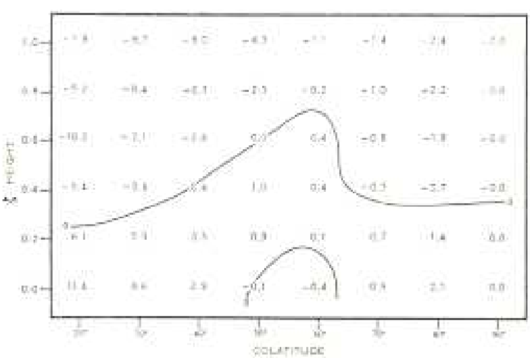
$\bar{u}$ m sec^{-1}

Figure 9. Zonal wind pattern of Experiment 2. Positive values denote westerly winds. Vertical ordinate is dimensionless height in range surface to tropopause.

value and vertical distribution of $\bar{u}$ agree with Tucker's (1964) observations. The polar easterlies extend further south than observed but the magnitudes are correct, suggesting strong polar winds. The surface zonal velocity reaches a realistic maximum of 3·5 m sec^{-1}.

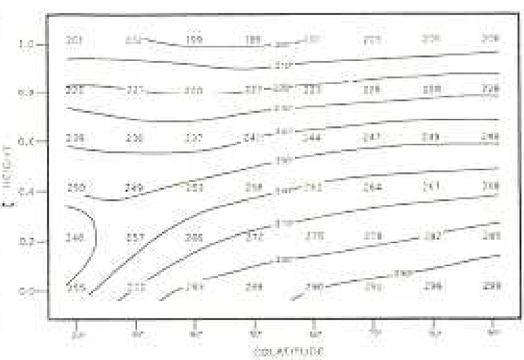
Although it is unnecessary to consider the series coefficients, since accurate total variables are produced, it is instructive to examine the structure of the dominant coefficients of the series. As an example, the terms of the $(\Delta T/T_0)^1$ series for v and u are illustrated schematically in Fig. 6. A comparison of magnitudes shows the initial two terms as being dominant with the higher P power terms diminishing in importance. We have seen that v is a Hadley cell so that V_{10} must dominate the higher powered multicellular systems in which the Hadley-Ferrel-Hadley tricellular pattern is prominent. The pattern of the zonal coefficients shows a tendency towards a breaking up in the higher P powers. These terms are characteristic of strong rotation in the same way that P° is characteristic of thermally dominated flows – Hadley cells are produced in P° for all $(\Delta T/T_0)$ powers. The same structures occur with only slight differences in the other experiments.

For Experiments 2, 3 and 4, a heat-balance estimate is used in which condensation-precipation differences have been allowed for. The heat flux is smaller than in the preceding experiment so that lower turbulence parameter values are needed.



$\bar{V}$ cm sec^{-1}

Figure 10. Meridional wind pattern of Experiment 3, indicating a tricellular system. Positive values indicate flow towards the Equator. Vertical ordinate is dimensionless height in range surface to tropopause.



T °K

Figure 11. Temperature field of Experiment 3. Vertical ordinate is dimensionless height in range surface to tropopause.

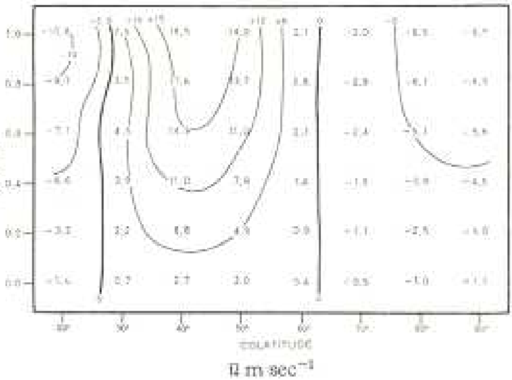


Figure 12. Zonal wind pattern of Experiment 3, with a jet maximum of 19 m sec^{-1}. Positive values denote westerly winds. Vertical ordinate is dimensionless height in range surface to tropopause.

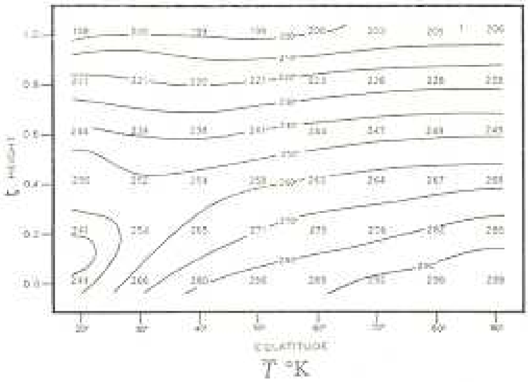


Figure 13. Temperature field of Experiment 4. Comparison with Fig. 11 illustrates the effect of a small surface exchange. Vertical ordinate is dimensionless height in range surface to tropopause.

The meridional velocity fields of these experiments (e.g. Figs. 7 and 10) also favour a single Hadley cell. This is due to the parameter values not being correctly specified to meet the atmospheric requirements. This was confirmed by other experiments where the tricellular pattern was produced but the $\bar{u}$, $\bar{T}$ variables were not as realistic as here. Isolation of a completely correct combination of parameters is difficult and was not achieved. The meridional velocity field of Experiment 3 does, however, contain a small mid-latitude Ferrel cell, indicating its possible presence.

The temperature fields are also realistic, as shown in Figs. 8, 11 and 13. Comparison between the fields of Experiments 2 and 3 shows that the latter and larger K_{MT} value reduces the Pole-Equator temperature difference and that taking $J(\zeta)$ as a vertically increasing function produces better polar temperature values. Further, the meridional gradients show that the distribution of $\overline{u'v'}$ is in keeping with the atmospheric pattern.

Experiment 4 is identical with Experiment 3 except for the inclusion of a small additional exchange function, Q_{S+L}. The $\bar{u}$, $\bar{v}$ fields are only slightly altered and are not illustrated. The most significant differences can be seen in the temperature fields, Figs. 11 and 13. The Q_{S+L} function produces lower temperatures in the upper and lower troposphere and higher values in the mid-troposphere. This suggests that the function is transferring heat upward but little can be deduced due to an artificial constraint on the exchange function.

The distributions produced for the zonal velocity are also realistic in sign and gross features (particularly that of Experiment 3), as shown in Figs. 9 and 12. In Experiment 2, calculated wind magnitudes are somewhat smaller than observed but the characteristic east-west-east structure is present. The westerly and equatorial easterly maxima of 13 m sec^{-1} and 5 m sec^{-1} respectively for Experiment 2 become a more realistic 19 m sec^{-1} and 7 m sec^{-1} for Experiment 3. This is mainly due to reducing the zonal 'friction' i.e. K_{ZV}. The jet-like structure of Fig. 12 has a form and magnitude similar to the observed seasonal mean. The polar easterlies are farther south and larger at 11 m sec^{-1} than they ought to be. End point values are, however, not expected to be very accurate.

7. Conclusions

Numerical experiments have been performed with a quasi-stationary mean flow model of the general circulation based on a modification of the austausch concept. The realism of the calculated results for $\bar{v}$, $\bar{u}$ and $\bar{T}$ indicates that the relations assumed for the mean eddy mechanisms must generally represent satisfactorily the main functions of the large-scale eddies. The assumption that cyclone-scale eddies release potential energy as dictated by Eq. (19) reproduces the atmospheric distribution of the lateral eddy stress. The reasonable nature of the results further suggests that the use of eddy coefficients to relate the eddy to the mean flow is suitable within its limitations to some studies of the general circulation.

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