On the Direction of the Eddy Momentum Flux in Baroclinic Instability

ISAAC M. HELD

Geophysical Fluid Dynamics Laboratory/NOAA, Princeton University, Princeton, NJ 08540

DAVID G. ANDREWS

Department of Atmospheric Physics, Clarendon Laboratory, Oxford University, Oxford, England

(Manuscript received 28 January 1983, in final form 13 May 1983)

ABSTRACT

The direction of the vertically-integrated horizontal eddy flux of momentum in linear baroclinically unstable modes is investigated in a number of cases where the basic flow contains horizontal, as well as vertical, shear. A general result is presented for slowly-growing modes on a flow with weak horizontal shear. Some special cases are described in which standard baroclinic instabilities of finite growth rate (for an internal jet, Eady's model, and a two-layer model) are perturbed by weak horizontal shear, and some computations for flows with large horizontal shear are also mentioned. A general rule emerging from these calculations is that for flows with horizontal jet structure of broader scale than the radius of deformation, the vertically-integrated momentum flux tends to be into the jet (or upgradient); while for jets narrower than the radius of deformation, momentum fluxes tend to be out of the jet (downgradient), even when the contribution of horizontal curvature to the basic state potential vorticity gradient is negligible. However, some exceptions to this general rule exist.

1. Introduction

In the analysis of the instability of atmospheric zonal flows to quasi-geostrophic disturbances, one often encounters unstable modes that are essentially baroclinic, deriving most of their energy from the potential energy of the basic state, but with structures modified somewhat by horizontal shears in the zonal wind. Among these modifications the tilt of constant phase lines with latitude is of particular interest, this tilt being identically zero for normal modes in a purely baroclinic problem. Few general results have been obtained that predict even the sign of this tilt or, equivalently, the direction of the horizontal eddy flux of momentum, given the form of the mean flow.

One simple result, described in Held (1975), can be found by a straightforward manipulation of the disturbance potential vorticity equation: if the mean potential vorticity gradient is positive at all heights at some latitude y_0 and if the vertical shear is nonpositive (non-negative) at y_0 on a flat lower (upper) boundary [i.e., if the Charney–Stern (1962) sufficient condition for stability is satisfied at this latitude], then an unstable wave must produce a divergence of the vertically integrated eddy flux of westerly momentum at y_0. As a specific example, to which we shall return briefly in Section 4, consider the following westerly internal jet on a beta-plane in a Boussinesq atmosphere:

$$\bar{u}(y, z) = \bar{u}_0 \operatorname{sech}^2\left(\frac{z}{H}\right) + \bar{u}_1 \operatorname{sech}^2\left(\frac{y}{L}\right), \quad (1.1)$$

where $\bar{u}_0$ and $\bar{u}_1$ are positive constants. (Our notation is standard throughout; in particular, an overbar refers to a zonally symmetric basic state or to a zonal average and a prime to the disturbance.) The mean potential vorticity gradient is, in general,

$$\frac{\partial \bar{q}}{\partial y} = \beta - \frac{\partial^2 \bar{u}}{\partial y^2} - f^2 \frac{\partial}{\partial z} \frac{1}{N^2} \frac{\partial \bar{u}}{\partial z}.$$

Setting N^2 = constant, one finds that $\partial\bar{q}/\partial y$ is positive for all z at the center of the jet ($y = 0$) provided that

$$\beta - \frac{2f^2\bar{u}_0}{3N^2H^2} + \frac{2\bar{u}_1}{L^2} > 0, \quad (1.2)$$

in which case there must be a divergence of the vertically integrated momentum flux at the center of the jet; see (2.4). Furthermore, one can easily show from symmetry that all modes have zero momentum flux at $y = 0$: since $\bar{u}$ is symmetric in y, all modes must be symmetric or antisymmetric about $y = 0$ and $u'v' = -\psi'_x\psi'_y$ must be antisymmetric. Therefore, the vertically integrated flux is directed out from the jet center if (1.2) is satisfied. One obtains no information about the direction of the momentum flux at $y = 0$ if (1.2) is not satisfied. For example, if the contribution of the horizontal curvature to $\partial\bar{q}/\partial y$ is small compared with $2f^2\bar{u}_0/(3N^2H^2) - \beta > 0$.

The case when (1.2) does not hold is essentially analogous to that which arises when a flow destabilized by a temperature gradient at a boundary is perturbed by horizontal shears, e.g.

$$\bar{u} = \Lambda z + \bar{u}_1 \operatorname{sech}^2\left(\frac{y}{L}\right), \quad z \geq 0.$$

Whatever the values of $\bar{u}_1$ and L, the barotropic contribution to $\partial\bar{q}/\partial y$ cannot alter the fact that the sufficient condition for stability is violated at all latitudes by the vertical shear at $z = 0$ (assuming $\Lambda > 0$). As a result, no information concerning the direction of momentum fluxes is obtained from the result described by Held (1975) in many atmospheric applications.

When the y variation of $\bar{u}$ creates only small perturbations in $\bar{u}$ and $\partial\bar{q}/\partial y$, one can develop a formal perturbation theory in which an unstable mode of the pure baroclinic problem is the lowest order solution. McIntyre (1970) considers this theory for Eady's problem perturbed by a zonal velocity field dependent only on y

$$\bar{u} = \Lambda z + \mu\bar{u}_1(y), \quad 0 \leq z \leq H,$$

where $\beta = 0$, $N^2 =$ constant, $\mu \ll 1$. When the characteristic horizontal scale of the zonal wind variations is much greater than the radius of deformation NH/f, McIntyre finds that the vertically integrated flux of westerly momentum is directed countergradient and, therefore, into the center of a westerly jet. The restriction to broad jets is essential for this result. The proof is sufficiently complex that it is unclear (to us) under what circumstances it can be generalized to perturbations of more general form, $\mu\bar{u}_1(y, z)$, or to unperturbed flows other than Eady's.

A number of detailed studies of particular flows have been described in the literature (e.g., Pedlosky, 1964; Brown, 1969; Hart, 1974; Holland and Haidvogel, 1980; Conrath *et al.* 1981). The variety of results emphasizes the difficulty of obtaining general results concerning the direction of the eddy momentum fluxes. In particular, Brown finds modes of relatively small zonal wavelength possessing a negative barotropic zonal-to-eddy energy conversion

$$-\iint \overline{u'v'}\partial\bar{u}/\partial y dy dz < 0,$$

and modes of relatively large wavelength possessing a positive conversion, both growing on the same zonal flow. However, the various calculations do support the rule that eddy momentum fluxes are countergradient when the jet is sufficiently broad. This rule is further supported by multiple-scale approximations for the eigenfunctions on broad jets, valid when $\bar{u}$ varies more slowly with y than the phase of the mode under consideration (Stone, 1969; Simmons, 1974; Gent, 1975; Killworth, 1981).

Is the rule that momentum fluxes are countergradient on broad jets valid in general and does it have a simple explanation? Is the flux countergradient when the jet is sufficiently broad that $\partial^2\bar{u}/\partial y^2 \ll \beta$ [as suggested in the concluding remarks of Pedlosky (1964) and Simmons (1974)], or is it also necessary that the meridional scale of the jet be larger than the appropriate deformation radius as implied by McIntyre (1970)? Conversely, is it typically the case that momentum fluxes are downgradient for jets that have meridional scales smaller than the deformation radius but that are sufficiently weak so that $\partial^2\bar{u}/\partial y^2 \ll \beta$? In the following, we first present a simple theorem and then a series of calculations for particular flows which, taken together, may help clarify some aspects of this problem. The theorem, described in Section 2, refers to the rather special situation in which: 1) the growth rate is sufficiently small that the meridional disturbance particle displacement field is localized about the steering level, and 2) the effects of horizontal shears on $\bar{u}$ and $\partial\bar{q}/\partial y$ are small. In Section 3, calculations are performed for internal jet, Eady, and two-layer models within a perturbation theory similar to McIntyre's for which growth rates need not be small. In order to simplify the calculations to the point that a large number can be performed and summarized, we consider only the special case in which the unperturbed baroclinic mode has no meridional structure. In Section 4, results are described for the particular flow given by 1), when the effects of horizontal shears on $\bar{u}$ and $\partial\bar{q}/\partial y$ are not small.

The results described here and in McIntyre (1970) and Held (1975) refer to the sign of the vertically integrated momentum flux, not to the sign of the momentum flux at a particular level. Fortunately, if one happens to be primarily concerned with the back effect of the amplifying disturbance on the zonal flow, then this vertically integrated flux is a quantity of prime interest. The acceleration of the mean zonal flow at a particular level is non-zero even in the pure baroclinic problem, due to the Coriolis force resulting from the mean meridional circulation associated with the eddy heat fluxes, but these accelerations average to zero in the vertical. Perturbing the flow with small horizontal shears produces small perturbations in these local accelerations, but the qualitatively new feature introduced is a non-zero vertically integrated momentum flux convergence. The tilt of constant phase lines at a particular level may still be of interest for other reasons, of course. For example, Hoskins and West (1979) suggest that the structure of fronts formed in a finite amplitude disturbance is related to the direction of the tilt of constant phase lines at the ground in the developing wave.

One might hope that results on the vertically integrated eddy momentum flux in quasi-geostrophic instabilities would help one understand the time-averaged, zonally-averaged surface winds, since the surface stresses produced by these winds must be balanced by the vertically integrated momentum fluxes. Unfortunately, the momentum fluxes averaged over the life cycle of baroclinic eddies often bear little resemblance to the fluxes during the growth stage of the disturbances (Simmons and Hoskins, 1978; Gall, 1976). Furthermore, Lau (1978) finds that the observed momentum fluxes peak further downstream in the oceanic storm tracks than the eddy heat fluxes, suggesting that the observed momentum fluxes arise primarily from mature, possibly decaying disturbances. Therefore, it must be admitted that linear instability analyses of the sort presented here may not contribute a great deal to our understanding of the climatological atmospheric momentum fluxes.

2. A result for weakly unstable jet instabilities

The discussion that follows is for the Boussinesq case, but the proof for a compressible atmosphere is similar.

Consider a mean zonal flow in which $\partial\bar{u}/\partial z = 0$ at lower and upper boundaries (if such boundaries exist), so that the eddy heat flux also vanishes at these boundaries. Or consider a mode that is of sufficiently small amplitude at the boundaries that these heat fluxes can be ignored in any case. Using the identity

$$\frac{\partial}{\partial y}\overline{u'v'} - f^2\frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'\partial\psi'/\partial z}\right) = -\overline{v'q'}, \quad (2.1)$$

(e.g., Green, 1970) one can relate the vertically integrated eddy momentum flux convergence to the potential vorticity flux:

$$\frac{\partial M}{\partial y} = -\int \overline{v'q'}dz, \quad (2.2)$$

where

$$M \equiv \int \overline{u'v'}dz \quad (2.3)$$

and vertical integrals are taken throughout the depth of the fluid. If the northward particle displacement field η' is defined so that

$$\frac{\partial\eta'}{\partial t} + \bar{u}\frac{\partial\eta'}{\partial x} = v'.$$

Then $q' = -\eta'\partial\bar{q}/\partial y$ for linearized disturbances in an inviscid adiabatic fluid and

$$\frac{\partial M}{\partial y} = \int \overline{\eta' v'}\frac{\partial\bar{q}}{\partial y}dz = \int K\frac{\partial\bar{q}}{\partial y}dz, \quad (2.4)$$

where $K(y, z, t) = \tfrac{1}{2}\partial\overline{\eta'^2}/\partial t$ is the rate of increase of the north–south dispersion of a set of particles centered on latitude y. For a linear unstable mode, $\eta' = \mathrm{Re}\,\eta(y, z)e^{ik(x-ct)}$, with zonal wavenumber $k > 0$ and complex phase speed $c(k) = c_R(k) + ic_I(k)$, one has $K(y, z, t; k) = kc_I|\eta|^2e^{2kc_It}$.

For the rest of this section we confine attention to the special case of slowly growing modes $c_I \downarrow 0$ (corresponding to $k \to k_N$, say) whose steering levels lie within the fluid. [The steering level of a mode is that height $Z(y)$ at which $\bar{u} = c_R$; it varies with latitude if $\bar{u}$ does so.] It can be shown [e.g., Bretherton, 1966, Eq. (17)] that K approaches a delta function at the steering level as $c_I \to 0$ from above (and $k \to k_N$):

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\eta'^2} \equiv K = n(y, t)\delta[z - Z(y)] + O(c_I), \quad (2.5)$$

where

$$n(y, t) = \left(\frac{k\pi\overline{\psi'^2}}{\left|\frac{\partial\bar{u}}{\partial z}\right|}\right)_{\substack{z=Z(y)\\k=k_N}} \geqslant 0.$$

From (2.4) and (2.5) it follows that

$$\frac{\partial M}{\partial y} = n(y, t)\frac{\partial\bar{q}}{\partial y}[y, Z(y)] + O(c_I), \text{ as } c_I \downarrow 0. \quad (2.6)$$

The sign of the vertically integrated momentum flux convergence is determined here by the sign of the mean potential vorticity gradient at the steering level.

We now suppose that the horizontal shear of the basic flow $\bar{u}$ is small, so that

$$\bar{u}(y, z) = \bar{u}_0(z) + \mu\bar{u}_1(y, z), \quad (2.7)$$

where $\mu \ll 1$. Likewise,

$$\frac{\partial\bar{q}}{\partial y} = \frac{\partial\bar{q}_0}{\partial y}(z) + \mu\frac{\partial\bar{q}_1}{\partial y}(y, z). \quad (2.8)$$

For a mode of phase speed $c_R(k)$, the steering level height $Z(y)$ satisfies

$$\begin{aligned}\bar{u}(y, Z(y)) &= \bar{u}_0(Z(y)) + \mu\bar{u}_1(y, Z(y))\\ &= c_R(k).\end{aligned} \quad (2.9)$$

We define Z_0 such that $\bar{u}_0(Z_0) = c_R(k)$ and put

$$Z(y) = Z_0 + \mu Z_1(y). \quad (2.10)$$

Substituting (2.10) in (2.9) and expanding, we obtain

$$Z_1\frac{\partial\bar{u}_0}{\partial z}(Z_0) + \bar{u}_1(y, Z_0) = O(\mu). \quad (2.11)$$

Also expanding $\partial\bar{q}/\partial y(y, Z(y))$ in powers of μ,

$$\frac{\partial\bar{q}}{\partial y}(y, Z(y)) = \frac{\partial\bar{q}_0}{\partial y}(Z_0) + \mu\frac{\partial^2\bar{q}_0}{\partial y\partial z}(Z_0)Z_1 + \mu\frac{\partial\bar{q}_1}{\partial y}(y, Z_0) + O(\mu^2).$$

Here Z_1 can be substituted from (2.11), giving

$$\frac{\partial\bar{q}}{\partial y}(y, Z(y)) = \frac{\partial\bar{q}_0}{\partial y}(Z_0) + \mu\left\{-\alpha(Z_0)\bar{u}_1(y, Z_0) + \frac{\partial\bar{q}_1}{\partial y}(y, Z_0)\right\} + O(\mu^2), \quad (2.12)$$

where

$$\alpha(z) \equiv \frac{\partial^2\bar{q}_0/\partial y\partial z}{\partial\bar{u}_0/\partial z}. \quad (2.13)$$

Eqs. (2.6) and (2.12) combine to yield

$$\frac{\partial M}{\partial y} = n(y, t)\frac{\partial\bar{q}_0}{\partial y}(Z_0) + \mu n(y, t) \times\left[-\alpha(Z_0)\bar{u}_1(y, Z_0) + \frac{\partial\bar{q}_1}{\partial y}(y, Z_0)\right] + O(\mu^2) + O(c_I). \quad (2.14)$$

Assuming that meridional velocities vanish on the walls of a channel or for sufficiently large $|y|$, we have $\int (\partial M/\partial y)dy = 0$ so that

$$0 = \frac{\partial\bar{q}_0}{\partial y}(Z_0)\int ndy - \mu\alpha(Z_0)\int n\bar{u}_1(y, Z_0)dy + \mu\int n\frac{\partial\bar{q}_1}{\partial y}(y, Z_0)dy + O(\mu^2) + O(c_I). \quad (2.15)$$

We define

$$\langle\alpha\rangle = \frac{\int n(y)a(y)dy}{\int n(y)dy},$$

for arbitrary $a(y)$, multiply (2.15) by $n(y)/\int ndy$ and subtract from (2.14); the term in $\partial\bar{q}_0/\partial y$ cancels, leaving

$$\frac{\partial M}{\partial y} = \mu n(y, t)\left[\frac{\partial\bar{q}_1}{\partial y} - \left\langle\frac{\partial\bar{q}_1}{\partial y}\right\rangle - \alpha(\bar{u}_1 - \langle\bar{u}_1\rangle)\right]_{z=Z_0} + O(\mu^2) + O(c_I). \quad (2.16)$$

It should be noted that $c(k)$ and $n(y, t)$ have not been expanded in μ since we want the limits $c_I \downarrow 0$ and $\mu \to 0$ to be independent. This is in contrast to the approach used by McIntyre (1970) and in Section 3 below.

As a simple example, suppose that N^2 is constant and $\bar{u}_1(y) = V\cos(ly)$ so that $\partial\bar{q}_1/\partial y = l^2V\cos(ly)$. Then from (2.16),

$$\frac{\partial M}{\partial y} \approx \mu n(l^2 - \alpha(Z_0))(\bar{u}_1 - \langle\bar{u}_1\rangle). \quad (2.17)$$

Suppose that we know the phase speed, and therefore $\alpha(Z_0)$ for a particular mode. Note that $n \geqslant 0$ and that $(\bar{u}_1 - \langle\bar{u}_1\rangle) > 0$ at the maxima of $\bar{u}_1$. Thus, for sufficiently narrow jets ($l^2 > \alpha$), $\partial M/\partial y$ is positive at the maxima of $\bar{u}_1$. (Note that these down-gradient fluxes are obtained within a perturbation theory which assumes that $\partial^2\bar{u}_1/\partial y^2 \ll \beta$.) If $\alpha(Z_0)$ is positive, then for sufficiently broad jets ($l < \alpha^{1/2}$) $\partial M/\partial y$ is negative. However, if $\alpha(Z_0)$ is negative, $\partial M/\partial y$ must be positive at the wind maxima for all values of l. That this latter situation does indeed occur is illustrated in Section 3 below. Thus, instabilities growing on purely baroclinic zonal flows perturbed by small horizontal shears need not possess countergradient momentum fluxes, even when the length scale of the y variation of the zonal wind is made arbitrarily large.

3. Examples of momentum fluxes associated with instabilities on zonal flows with small horizontal shears

a. *The perturbation theory*

The result (2.16) described above unfortunately holds only for a very restricted set of modes, namely slowly-growing disturbances in weak horizontal shear. It is conceivable, however, that $\partial M/\partial y$ behaves in a similar way even when the derivation of (2.16) is not strictly justified. To check on this possibility, we compute the momentum fluxes in modes growing on a particular internal jet flow for which c_I is not small, working with the μ-perturbation theory used by McIntyre, which differs from that described in Section 2 in that the disturbance streamfunction and phase speed, as well as the basic flow parameters, are expanded in μ. We also describe analogous calculations for the Eady and two-layer models. To simplify the analysis we consider only the case in which the unperturbed baroclinic mode is independent of y on an infinite beta-plane.

The equation to be solved for the complex streamfunction ψ and phase speed c is

$$(\bar{u} - c)\left(-k^2\psi + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\epsilon\frac{\partial\psi}{\partial z}\right) + \psi\partial\bar{q}/\partial y = 0, \quad (3.1a)$$

with boundary conditions

$$(\bar{u} - c)\frac{\partial\psi}{\partial z} = \psi\frac{\partial\bar{u}}{\partial z}, \quad (3.1b)$$

on boundaries $z = z_B$ for example; here $\epsilon(z) = f^2/N^2(z)$. We expand in μ:

$$\left.\begin{aligned} \bar{u} &= \bar{u}_0(z) + \mu\bar{u}_1(y, z) \\ \frac{\partial\bar{q}}{\partial y} &= \frac{\partial\bar{q}_0}{\partial y}(z) + \mu\frac{\partial\bar{q}_1}{\partial y}(y, z) \\ \psi &= \psi_0(z) + \mu\psi_1(y, z) \\ c &= c_0 + \mu c_1 \end{aligned}\right\}, \quad (3.2)$$

where the $O(\mu^0)$ terms have been chosen to correspond to a standard pure baroclinic eigenvalue problem. At $O(\mu)$ we can eliminate c_1 by differentiation with respect to y (since ψ_0, $\bar{u}_0$ and $\partial\bar{q}_0/\partial y$ depend only on z) to obtain

$$(\bar{u}_0 - c_0)\left(-k^2\phi + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial}{\partial z}\epsilon\frac{\partial\phi}{\partial z}\right) + \phi\frac{\partial\bar{q}_0}{\partial y} = -\frac{\partial\bar{u}_1}{\partial y}\left(-k^2\psi_0 + \frac{\partial}{\partial z}\epsilon\frac{\partial\psi_0}{\partial z}\right) - \psi_0\frac{\partial^2\bar{q}_1}{\partial y^2}, \quad (3.3a)$$

$$(\bar{u}_0 - c_0)\frac{\partial\phi}{\partial z} - \phi\frac{\partial\bar{u}_0}{\partial z} = -\frac{\partial\bar{u}_1}{\partial y}\frac{\partial\psi_0}{\partial z} + \psi_0\frac{\partial^2\bar{u}_1}{\partial y\partial z} \quad \text{on} \quad z = z_B, \quad (3.3b)$$

where

$$\phi(y, z) \equiv \frac{\partial\psi_1}{\partial y}. \quad (3.4)$$

Since all coefficients of ϕ in (3.3) are independent of y, solutions for ϕ can be found by separation of variables. For simplicity we take $\bar{u}_1(y, z) = U(y)g(z)$ (although similar methods can be applied to more general forms of $\bar{u}_1$) and Fourier transform in y:

$$U(y) = \int_{-\infty}^{\infty} \mathcal{U}(l)e^{ily}dl. \quad (3.5)$$

Inspection of (3.3) then shows that we can write

$$\phi(y, z) = \int_{-\infty}^{\infty} il\varphi(z; k, l)\mathcal{U}(l)e^{ily}dl, \quad (3.6)$$

where φ satisfies

$$(\bar{u}_0 - c_0)\left(-[k^2 + l^2]\varphi + \frac{\partial}{\partial z}\epsilon\frac{\partial\varphi}{\partial z}\right) + \varphi\partial\bar{q}_0/\partial y = -g\left(-k^2\psi_0 + \frac{\partial}{\partial z}\epsilon\frac{\partial\psi_0}{\partial z}\right) + \psi_0\left(-l^2g + \frac{\partial}{\partial z}\epsilon\frac{\partial g}{\partial z}\right), \quad (3.7a)$$

$$(\bar{u}_0 - c_0)\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial\bar{u}_0}{\partial z} = \psi_0\frac{\partial g}{\partial z} - g\frac{\partial\psi_0}{\partial z} \quad \text{on} \quad z = z_B. \quad (3.7b)$$

The momentum flux divergence can be written

$$\frac{\partial}{\partial y}\overline{u'v'} = -\frac{k\sigma}{2}\mathrm{Im}\left(\psi^*\frac{\partial^2\psi}{\partial y^2}\right),$$

where $\sigma \equiv \exp(2kc_it)$. Expanding in μ and using the fact that $\partial\psi_0/\partial y \equiv 0$, we find

$$\frac{\partial}{\partial y}\overline{u'v'} = -\tfrac{1}{2}\mu\sigma k\,\mathrm{Im}\left(\psi_0^*\frac{\partial^2\psi_1}{\partial y_2}\right) + O(\mu^2)$$
$$= \tfrac{1}{2}\mu\sigma k\,\mathrm{Im}\psi_0^*(z) \times \int_{-\infty}^{\infty} l^2\varphi(z; k, l)\mathcal{U}(l)e^{ily}dl + O(\mu^2) \quad (3.8)$$

by (3.4) and (3.6). Restricting attention to jets that are symmetric in y, $\bar{u}_1(y, z) = \bar{u}_1(-y, z)$ [so that $\mathcal{U}(l)$ is real and an even function of l] and also observing from (3.7a) that $\varphi(l) = \varphi(-l)$, we obtain from (2.3) and (3.8)

$$\frac{\partial M}{\partial y} \approx \mu\sigma\int_0^{\infty} \mathcal{U}(l)m(k, l)\cos(ly)dl, \quad (3.9)$$

where

$$m(k, l) = \int \mathrm{Im}[kl^2\psi_0^*(z; k)\varphi(z; k, l)]dz. \quad (3.10)$$

The special case $\bar{u}_1 = U\cos(ly)$ (U = positive constant) has westerly jets at $y = 2\pi n/l$ (n = integer); the vertically-integrated flux M is directed into the centers of these jets ("upgradient" momentum transfer) if $m(k, l) < 0$, and out of the jets ("downgradient") if $m(k, l) > 0$. We now proceed to calculate $m(k, l)$ for baroclinically unstable waves in a variety of circumstances.

b. *An internal jet*

First consider the basic flow that varies only with height

$$\bar{u}_0 = U_0\,\mathrm{sech}^2\left(\frac{z}{H}\right),$$

$$U_0 > 0, \quad N^2 = \text{constant},$$

for which

$$\frac{1}{\beta}\frac{\partial\bar{q}_0}{\partial y} = 1 + 2\tilde{U}\left[3\,\mathrm{sech}^4\left(\frac{z}{H}\right) - 2\,\mathrm{sech}^2\left(\frac{z}{H}\right)\right], \quad (3.12)$$

where

$$\tilde{U} \equiv \frac{U_0}{\beta\lambda^2}, \quad \lambda \equiv NH/f \equiv H\epsilon^{-1/2}.$$

This satisfies the Charney–Stern necessary condition for instability if $\tilde{U} > 1.5$. Note that $\alpha(z)$, defined by (2.13) is here given by

$$\alpha(z) = 4\lambda^{-2}\left[3\,\mathrm{sech}^2\left(\frac{z}{H}\right) - 1\right] \quad (3.13)$$

and thus takes either sign.

Eigenfunctions and eigenvalues for y-independent modes $\psi_0(z)$ that are symmetric about $z = 0$ are found by a combination of integrations of the initial value problem and an iterative shooting method. The assumption of symmetry about $z = 0$ allows one to

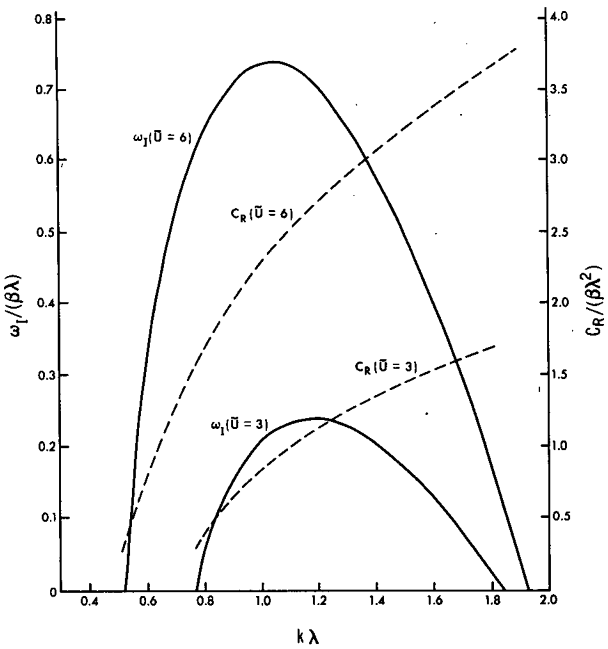

FIG. 1. Growth rates (ω_I) and phase speeds (c_R) as a function of zonal wavenumber for the unstable mode symmetric about $z = 0$ on the basic state defined in (3.11), for $\tilde{U} \equiv U_0/(\beta\lambda^2) = 3$ and 6.

place the lower boundary at $z = 0$, while the upper boundary is taken so high that the solutions are insensitive to its location.

Growth rates and phase speeds for these modes are shown in Fig. 1, for $\tilde{U} = 3$ and 6. Analytical symmetric neutral solutions $\tilde{\psi}_0 \propto \operatorname{sech}^2(z/H)$ exist at the wavenumbers $k\lambda = [2 \pm 2(1 - 3/2\tilde{U}))^{1/2}]^{1/2}$, just as for the classical Bickley jet (cf., Drazin and Howard, 1966); these correspond to the cutoff wavenumbers ($k\lambda = 0.76$, 1.85 for $\tilde{U} = 3$; $k\lambda = 0.52$, 1.93 for $\tilde{U} = 6$) at which $\omega_I \to 0$ in Fig. 1. The vertical structure of $\overline{\eta_0'^2}$ and $\overline{v_0'q_0'}$ for one of the more unstable modes with $\tilde{U} = 3$ ($k\lambda = 1.18$) is shown in Fig. 2, using the normalization $\int \overline{(\eta_0'/\lambda)^2} d(z/H) = 1$. $\overline{\eta_0'^2}$ has a reasonably sharp peak at the steering level ($z = 1.1H$), but its value at $z = 0$ is $\sim$25% of the peak value. Positive values of $\overline{v_0'q_0'}$ are confined fairly close to the steering level, but negative values peak at $z = 0$. Thus, it is not self-evident that the qualitative behavior of the momentum fluxes obtained when this mode is perturbed will be similar to that described in Section 2, in which $\overline{v_0'q_0'}$ is assumed localized about the steering level.

We consider two different perturbations about the flow (3.11):

$$1) \quad \bar{u}(y, z) = U_0 \operatorname{sech}^2\left(\frac{z}{H}\right) + \mu U_1(y),$$

$$2) \quad \bar{u}(y, z) = U_0 \operatorname{sech}^2\left(\frac{z}{H}\right) + \mu U_1(y) \operatorname{sech}^2\left(\frac{z}{H}\right).$$

These both contain weak horizontal shears, as in Section 3a, with $g(z) \equiv 1$ in 1), and $g(z) \equiv \operatorname{sech}^2(z)$ in 2). We calculate $m(k, l)$ by solving (3.7), given each previously determined $\psi_0(z)$ and the appropriate $g(z)$ and substituting in the integral (3.10). Note that the sign of m is independent of the normalization, $\int \overline{(\eta_0'/\lambda)^2} d(z/H) = 1$.

The solution to (3.7) is sensitive to small errors in the unperturbed eigenfunction ψ_0 and eigenvalue c_0 as $l \to 0$, a consequence of ψ_0 being an eigenfunction of

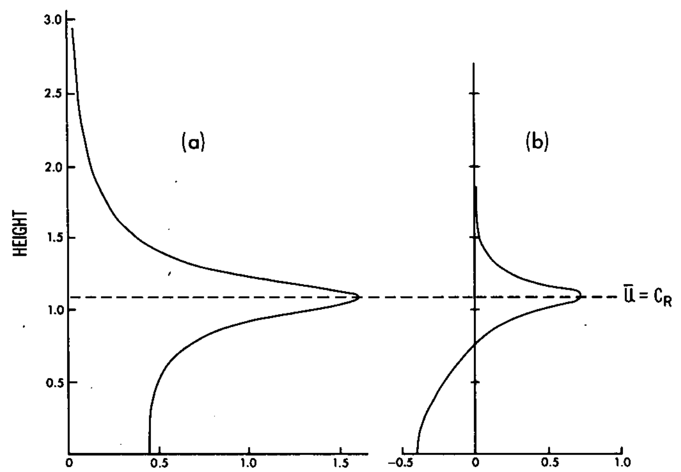

FIG. 2. The vertical structure of (a) $\overline{\eta'^2}$, and (b) $\overline{v'q'}$ for the unstable mode symmetric about $z = 0$ on the basic state defined in (3.11), for $\tilde{U} = 3$ and $k\lambda = 1.18$.

the operator on the left-hand side of (3.7a) with eigenvalue $-l^2$. It is therefore convenient to have an explicit expression for $m(k, l)$ in this limit. In the Appendix we show for an internal jet that

$$\lim_{l\to 0} m(k, l) = -k\gamma_1 \operatorname{Im}\gamma_2^{-1} \times \int dz\left[\frac{\psi_0}{\bar{u}_0 - c_0}\left(\psi_0 \frac{\partial}{\partial z}\epsilon\frac{\partial g}{\partial z} - q_0 g\right)\right], \quad (3.14)$$

where

$$\left.\begin{aligned} \gamma_1 &\equiv \int dz|\psi_0|^2 \\ \gamma_2 &\equiv \int dz\psi_0^2 \\ q_0 &\equiv -k^2\psi_0 + \frac{\partial}{\partial z}\epsilon\frac{\partial\psi_0}{\partial z} \end{aligned}\right\}.$$

This result has been used in drawing the figures which follow.

The values of $m(k, l)$ for case 1) with $\tilde{U} = 3$ are shown in Fig. 3a; the region where $m > 0$ has been shaded. m is positive for all l when k is near the longwave cutoff. For larger k there exists a critical value of l, $l_{\text{crit}}(k)$, such that $m \gtrless 0$ for $l \gtrless l_{\text{crit}}$.

The values of $m(k, l)$ for case 2) with $\tilde{U} = 3$ are plotted in Fig. 3b. Unlike case 1), $m < 0$ for sufficiently small l at all k. Otherwise, the results are qualitatively similar; in particular, $\partial l_{\text{crit}}/\partial k > 0$ in both cases. As a result, for perturbation zonal winds of the form $U_1(y) = \cos(ly)$, with l within a certain range [$l\lambda < 1.67$ in 1) and $0.76 < l\lambda < 1.85$ in 2]] disturbances of shorter zonal wavelength transport eddy momentum upgradient while those of longer zonal wavelength transport eddy momentum downgradient.

At $y = 0$ one has

$$\frac{\partial M}{\partial y} \approx \mu\sigma \int_0^\infty \mathcal{U}(l)m(k, l)dl,$$

where for prototypical westerly jets such as $U_1(y) = \exp[-(y/L)^2]$ or $U_1(y) = \text{sech}^2(y/L)$, $\mathcal{U}(l)$ is positive, nearly constant for $l \ll L^{-1}$, and nearly zero for $l \gg L^{-1}$. Therefore, if there exists an l_{crit} as defined above, the vertically integrated eddy momentum flux is directed out of sufficiently narrow jets ($L \ll l_{\text{crit}}^{-1}$) and into sufficiently broad jets ($L \gg l_{\text{crit}}^{-1}$). If $m > 0$ for all l, as in case 1) for $k\lambda < 1.06$, the momentum flux is directed out of the jet center, regardless of L.

Figs. 4a and 4b show how $l_{\text{crit}}(k)$ varies as $\tilde{U}$, a measure of the vertical shear of the unperturbed wind, is increased from 3 to 6. In both 1) and 2) l_{crit} increases with $\tilde{U}$, favoring upgradient over downgradient fluxes as the flow becomes more baroclinically unstable.

At the cutoff wavenumbers of the zero-order problem, these results can be checked against those of Section 2 (which hold in the limit $c_I \downarrow 0$) for the case $\bar{u}_1 = g(z)\cos(ly)$. The corresponding $\partial\psi_1/\partial y$ is proportional to $\cos(ly)$, and it can be shown that $c_1 \equiv 0$ for this example. At the cutoff, $c_{I,0} = 0$ and so $c_I = O(\mu^2)$; moreover, $Z_0 = z_0 + O(\mu^2)$ where $\bar{u}_0(z_0) = c_{R,0}$. Thus we obtain from (2.16)

$$\frac{\partial M}{\partial y} = \mu n_0\sigma\left\{\frac{\partial\bar{q}_1}{\partial y} - \left\langle\frac{\partial\bar{q}_1}{\partial y}\right\rangle - \alpha(\bar{u}_1 - \langle\bar{u}_1\rangle)\right\}_{z_0} + O(\mu^2)$$

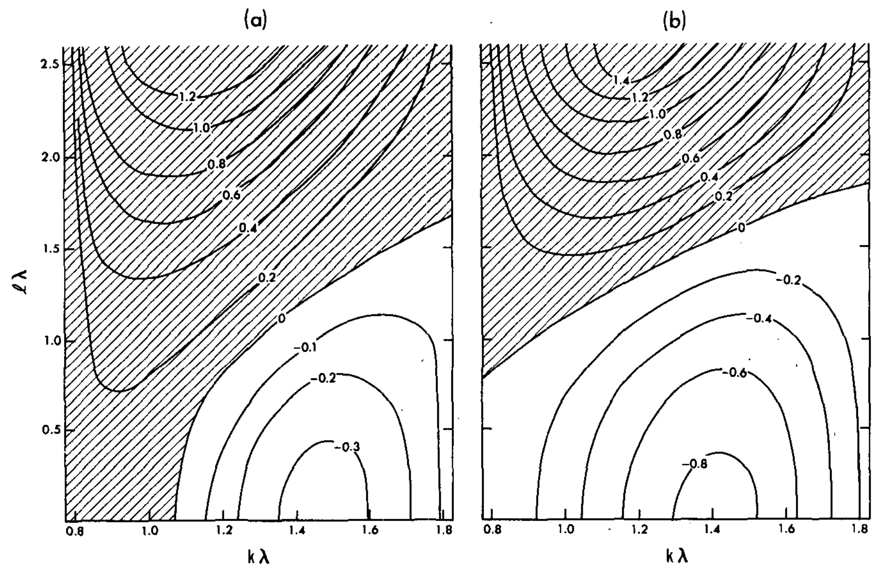

FIG. 3. $m(k, l)$ for the internal jet with $\tilde{U} = 3$: (a) $g(z) = 1$; (b) $g(z) = \text{sech}^2(z/H)$. The region in which $m > 0$ is shaded.

at the cutoffs, with n_0 a constant. (Note that if $c_I = O(\mu)$ instead of $O(\mu^2)$ additional $O(\mu)$ terms would appear in $\partial M/\partial y$.) Since $\bar{u}_1$ and $\partial\bar{q}_1/\partial y$ are sinusoidal in y,

$$\frac{\partial M}{\partial y} \approx \mu n_0 \sigma \left[l^2 g - \frac{\partial}{\partial z}\left(\epsilon \frac{\partial g}{\partial z}\right) - \alpha g \right]_{z_0} \cos(ly); \quad (3.15)$$

from (3.5) and (3.9) we also have

$$\frac{\partial M}{\partial y} \approx {}^1\!/\!_2 \sigma \mu m \cos(ly), \quad (3.16)$$

and so

$$m = 2n_0 \left[(l^2 - \alpha) g - \frac{\partial}{\partial z}\left(\epsilon \frac{\partial g}{\partial z}\right) \right]_{z_0} \quad (3.17)$$

by comparison. Therefore,

$$l_{\mathrm{crit}}^2 = \left[\alpha + \frac{1}{g}\frac{\partial}{\partial z}\left(\epsilon \frac{\partial g}{\partial z}\right) \right]_{z_0} \quad (3.18)$$

at the cutoffs. One can find z_0 by using the fact that $\partial\bar{q}_0/\partial y(z_0) = 0$ [which follows by letting $\mu \to 0$ in (2.15)]; then from (3.12)

$$\operatorname{sech}^2\left(\frac{z_0}{H}\right) = \frac{1}{3}\left[1 \pm \left(1 - \frac{3}{2\tilde{U}}\right)^{1/2}\right]$$

at the cutoff wavenumbers

$$k_\pm \lambda = \left[2 \pm 2\left(1 - \frac{3}{2\tilde{U}}\right)^{1/2}\right]^{1/2},$$

from which it is straightforward to calculate $\alpha(Z_0)$ and l_{crit}. For $g(z) \equiv 1$, one finds that $(\lambda l_{\mathrm{crit}})^2 = 2[(k\lambda)^2 - 2]$ at the short wave cutoff, while $l_{\mathrm{crit}}^2 < 0$ at the longwave cutoff (see Fig. 4a). This is consistent with the result that $m > 0$ for all l near the long wave cutoff. For $g(z) = \operatorname{sech}^2(z/H)$, one finds simply that $l_{\mathrm{crit}} = k$ at both short and long wave cutoffs, as illustrated in Fig. 4b.

c. Eady's model

Because of the presence of mean vertical shear (and the associated eddy heat flux) at the lower and upper boundaries, the argument of Section 2 cannot be directly applied to Eady's model perturbed by weak horizontal shear. We therefore proceed with the explicit perturbation theory to investigate whether the behavior of the momentum fluxes for this case is qualitatively similar to that for the internal jet.

The results for $m(k, l)$ for the perturbed Eady model

1) $\bar{u}(y, z) = \Lambda z + \mu U_1(y), \quad 0 \leq z \leq H$

$(\beta = 0, N^2 = \text{constant}, \Lambda > 0),$

are plotted in Fig. 5. The transition from upgradient flux for $l < l_{\mathrm{crit}}(k)$ to downgradient flux for $l > l_{\mathrm{crit}}(k)$ is again observed. Unlike the internal jet, however, the flux is upgradient for sufficiently small l at all k, in agreement with McIntyre's result.

If Eady's model is perturbed into the form

2) $\bar{u}(y, z) = \Lambda z + \mu U_1(y)\left(\frac{z}{H} - \frac{1}{2}\right),$

it is found that $m(k, l) \equiv 0$. The proof uses the facts that $\bar{u}_0 - c_0$ is symmetric about $z = H/2$ (recall that $c_{R,0} = \Lambda H/2$ for the unstable Eady mode) and that

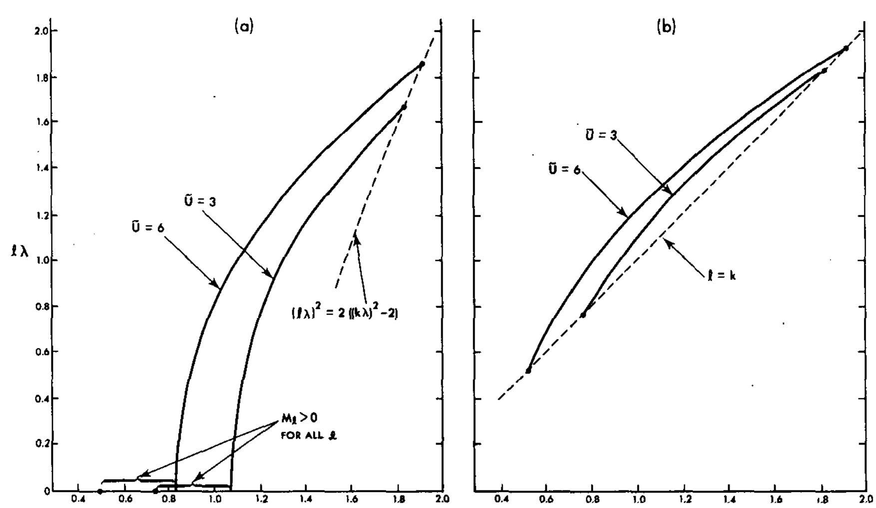

FIG. 4. $l_{\mathrm{crit}}(k)$ for the internal jet with $\tilde{U} = 3$ and $\tilde{U} = 6$: (a) $g(z) = 1$; (b) $g(z) = \operatorname{sech}^2(z/H)$. In case (a), $(\lambda l_{\mathrm{crit}})^2 = 2[(\lambda k)^2 - 2]$ at the short wave cutoff; in case (b), $l_{\mathrm{crit}} = k$ at both short and long wave cutoffs.

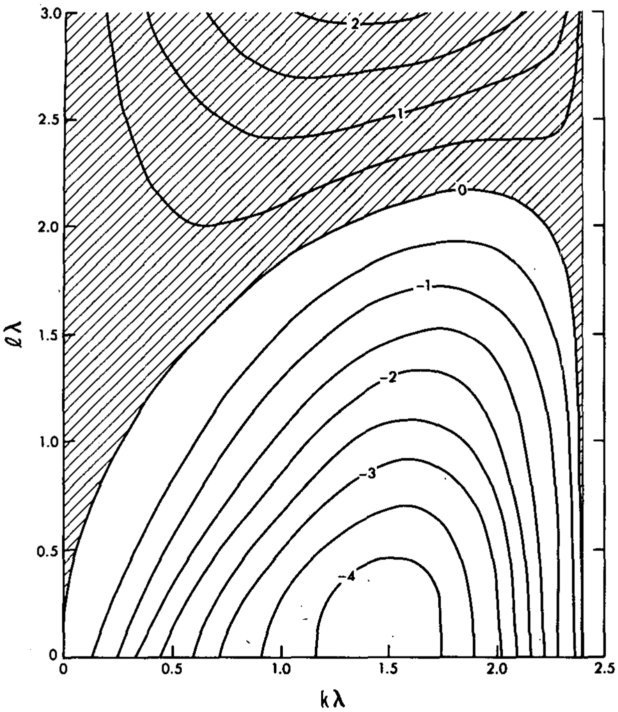

FIG. 5. $m(k, l)$ for Eady's model, with $g(z) \equiv 1$. The region in which $m > 0$ is shaded.

$\partial\bar{q}_0/\partial y \equiv 0$. Moreover, ψ_0 can be normalized such that $\psi_0(z) = \psi_0^*(H - z)$ [McIntyre, 1970, Eq. (A9)]; with $g(z) = z/H - ½$ it then follows from (3.7) that $\varphi(z) = \varphi^*(H - z)$, so that $\mathrm{Im}(\psi_0^*\varphi)$ is antisymmetric about $z = H/2$ and $m(k, l) = 0$ by (3.10).

The value of $m(k, l)$ for a perturbed wind of the form

3) $$\bar{u}(y, z) = \Lambda z + \mu U_1(y)z/H,$$

can be obtained by taking a linear combination of the results from 1) and 2), since φ, and therefore m, is linear in $g(z)$ by (3.7). Thus, in this case m takes values proportional to those in Fig. 5.

d. The two-layer model

Calculations have also been performed for the quasi-geostrophic two-layer model with layers of equal mean depth,

$$\left.\begin{aligned} \frac{\partial q_i'}{\partial t} &= -\bar{u}_i\frac{\partial q_i'}{\partial x} - \frac{\partial \psi_i'}{\partial x}\frac{\partial \bar{q}_1}{\partial y}, \quad i = 1, 2 \\ q_1' &= \frac{\partial^2 \psi_1'}{\partial y^2} - k^2\psi_1' - \frac{1}{2\lambda^2}(\psi_1' - \psi_2') \\ q_2' &= \frac{\partial^2 \psi_2'}{\partial y^2} - k^2\psi_2' - \frac{1}{2\lambda^2}(\psi_2' - \psi_1') \end{aligned}\right\},$$

where suffixes 1 and 2 refer to the upper and lower layers, respectively. The unperturbed problem is that of Phillips (1951), so that

$$\left.\begin{aligned} \bar{u}_1(y) &= U_0 + \mu U_1(y)g_1 \\ \bar{u}_2(y) &= -U_0 + \mu U_1(y)g_2 \end{aligned}\right\},$$

for U_0, g_1, g_2 constants. A theory analogous to that of Section 3a can be developed for $m(k, l)$, and it is convenient to consider pure "barotropic" ($g_1 = g_2$) and "baroclinic" ($g_1 = -g_2$) mean flow perturbations separately. When $\beta = 0$, m vanishes identically for a purely baroclinic perturbation, as in case 2) of the Eady model, while $m(k, l)$ for a barotropic perturbation, shown in Fig. 6, is similar to that in Fig. 5 for case 1) of Eady's model, except near the short-wave cutoff. The behavior of the perturbed Eady model momentum fluxes near the short-wave cutoff can be shown to result from the non-zero interior potential vorticity gradients of the perturbed flow, which have no analogue in the two-layer model. Straightforward algebra yields an explicit expression for l_{crit} in Fig. 6:

$$(\lambda l_{\mathrm{crit}})^2 = ½[(8\zeta^2 + 9\zeta^4)^{1/2} - \zeta^2],$$

where $\zeta \equiv k\lambda$.

Fig. 7 shows $m(k, l)$ for barotropic and baroclinic perturbations, for $\beta\lambda^2/U_0 = 0.5$. The barotropic jet perturbation results in momentum fluxes analogous to those in the internal jet of Fig. 3b, with $m > 0$ for

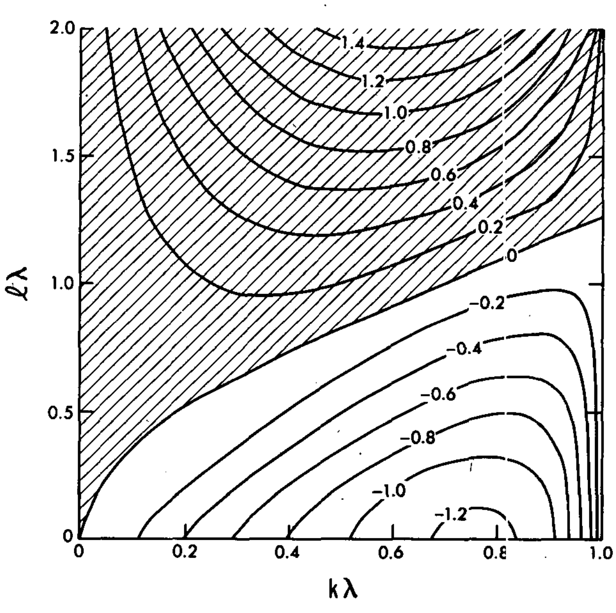

FIG. 6. $m(k, l)$ for the two-layer model with $\beta = 0$ and $g_1 = g_2 = 1$. The region in which $m > 0$ is shaded. The unperturbed eigenfunctions have been normalized so that $½|\psi_{0,1} + \psi_{0,2}| = U_0\lambda$.

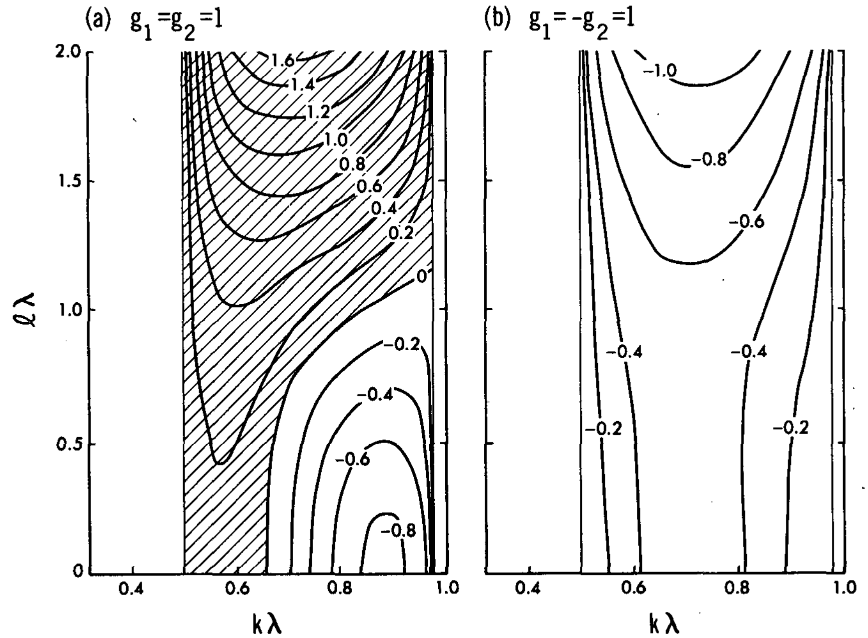

FIG. 7. $m(k, l)$ for the two-layer model with $\beta\lambda^2/u_0 = 0.5$. (a) $g_1 = g_2 = 1$; (b) $g_1 = -g_2 = 1$.

all l when k is close to the long-wave cutoff. The purely baroclinic jet perturbation results in $m < 0$ for all k and l, corresponding to a vertically integrated momentum flux convergence at the latitudes of the upper level wind maxima—a result for which we have no simple explanation.

4. A flow with large horizontal shears

Consider once again the flow given by (1.1) on a beta-plane in a Boussinesq atmosphere with constant N^2, choosing in particular $\tilde{U} = \bar{u}_0/(\beta\lambda^2) = 3$, where $\lambda \equiv NH/f$. From (1.2) we know for this value of $\tilde{U}$ that if $L^{-2} > \beta/(2\bar{u}_1)$, then $\partial\bar{q}/\partial y > 0$ for all z at $y = 0$, and, therefore, $\partial M/\partial y$ at $y = 0$ must also be positive. But $\beta/(2\bar{u}_1)$ is clearly an overestimate of the value of L^{-2} needed to create positive $\partial M/\partial y$ at $y = 0$, given $\bar{u}_1$; as we have seen, it is sufficient that $\partial\bar{q}/\partial y$ be positive at those heights where $\overline{\eta'^2}$ is large.

Fixing $\tilde{U} = 3$, we isolate the most unstable mode at one particular wavenumber, $k\lambda = 1.1$, for various values of $\bar{u}_1$ and L by a numerical integration of the initial value problem. Rigid walls are placed at $z = 0$, $4H$ and $y = \pm 4\pi\lambda$, and only modes symmetric about $z = 0$ are considered. The sign of this unstable mode's vertically integrated momentum flux at $y = 0$ is displayed in Fig. 8 as a function of L^{-1} and $\bar{u}_1$. As $\bar{u}_1 \to 0$, there is a critical value of L^{-1}, $\lambda L_{\text{crit}}^{-1} \approx 0.5$, above which $\partial M/\partial y > 0$ and below which $\partial M/\partial y < 0$. This result is similar to that of Section 3 (Fig. 4) despite the fact that those results are not strictly applicable here: in Section 3 we assume that the unperturbed baroclinic mode has infinite meridional scale; here the mode is confined to a channel of finite width. As $\bar{u}_1$ increases L_{crit}^{-1} decreases, which is the qualitative behavior one anticipates from (1.2). At least up to $\bar{u}_1 \approx 1.5\beta\lambda^2$, however, $(\beta/2\bar{u}_1)^{1/2}$ remains a very significant overestimate of L_{crit}^{-1}. Fig. 8 emphasizes once again that even for jets for which $\partial^2\bar{u}/\partial y^2 \ll \beta$ one finds downgradient momentum fluxes when the horizontal scale of the jet is smaller than the deformation radius.

5. Conclusions

The divergence of the vertically integrated momentum flux produced by an internal jet instability in the limit of vanishing growth rate is of the same sign as the mean potential vorticity gradient at the steering level. If in addition the mean flow takes the form

$$\bar{u}(y, z) = \bar{u}_0(z) + \mu\bar{u}_1(y, z)$$

with $\mu \ll 1$, then $\partial M/\partial y$ can be shown to depend on the parameter $\alpha(Z_0)$, where

$$\alpha(z) \equiv \frac{\partial^2\bar{q}_0/\partial y\partial z}{\partial\bar{u}_0/\partial z} = -\frac{\partial^3\bar{u}_0/\partial z^3}{\partial\bar{u}_0/\partial z}$$

and where $\bar{u}_0(Z_0) = c_R$, the phase speed of the mode in question. If $\bar{u}_1$ takes the simple form $\sin(ly)$, then the vertically integrated momentum flux is countergradient if $l^2 < \alpha$ and downgradient if $l^2 > \alpha$. If $\alpha < 0$, as it is for the jet $\bar{u}_0(z) = U_0 \operatorname{sech}^2(z/H)$ near the long-wave cutoff, the flux is downgradient for all l.

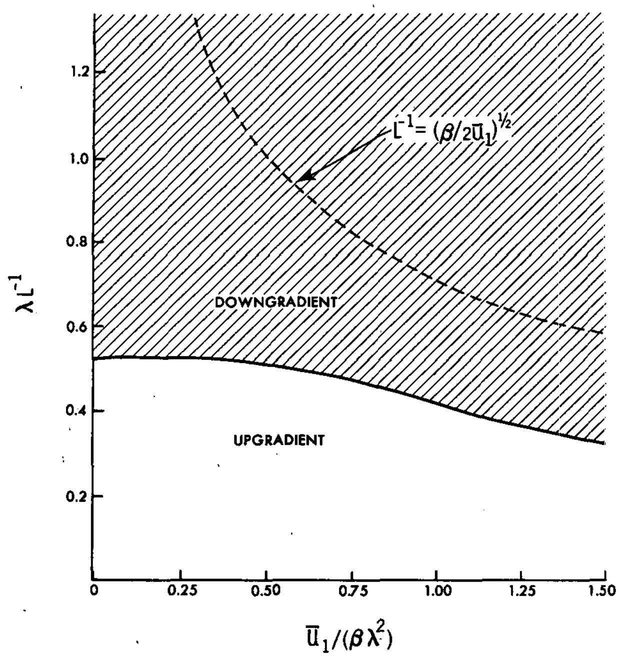

FIG. 8. The sign of the vertically integrated eddy momentum flux divergence at $y = 0$ for the most unstable wave on the basic state (1.1) with $\bar{u}_0/(\beta\lambda^2) = 3$ and $k\lambda = 1.1$, as a function of $\bar{u}_1$ and L^{-1}. There is divergence in the shaded region and convergence in the unshaded region. The dotted line marks that value of L^{-1} above which $\partial\bar{q}/\partial y$ at $y = 0$ is positive at all heights.

Since α is proportional to the *third* derivative of $\bar{u}_0$, one expects momentum fluxes to be very sensitive to small changes in the vertical structure of the mean flow.

Results from a perturbation theory in the small parameter μ indicate that the qualitative dependence of the momentum flux on the meridional scale of the mean flow-variations found for internal jet instabilities with small growth rates remains the same when growth rates are not small. Similar calculations for Eady and two-layer models show several differences in detail from the internal jet but also some common features. In particular, if one restricts consideration to the zonal wavenumbers corresponding to the fastest growing modes, then in each case there is a transition from upgradient to downgradient momentum fluxes as the meridional scale of the mean flow decreases. The transition occurs even when $\partial^2\bar{u}/\partial y^2 \ll \beta$, and at a scale comparable to the relevant radius of deformation or, equivalently, the zonal scale of the mode.

One can imagine perturbing the statistically steady state of a baroclinically unstable flow with a barotropic zonal jet. On the basis of the results described above, we expect this jet to deform the baroclinic instabilities present so as to produce vertically averaged momentum fluxes which enhance the jet if its meridional scale is much larger than the radius of deformation and dissipate the jet if its scale is much smaller than the radius of deformation.

Acknowledgments. The authors would like to thank B. J. Hoskins for several helpful discussions on this topic and Y. Hayashi for a review of the manuscript. One of the authors (DGA) acknowledges support at Princeton University from NOAA Grant 04-7-022-44017, and at Oxford from a Royal Society Meteorological Office Research Fellowship.

APPENDIX

Calculation of $m(k, l)$ as $l \to 0$

We consider the special case of an internal jet, but the result is easily generalized to problems with boundaries.

Write (3.7a) in the form

$$\frac{\partial}{\partial z}\left(\epsilon\frac{\partial\varphi}{\partial z}\right) + \left(\frac{\partial\bar{q}_0/\partial y}{\bar{u}_0 - c_0} - k^2 - l^2\right)\varphi = F, \quad \text{(A1)}$$

where F is the rhs of (3.7a) divided by $(\bar{u}_0 - c_0)$. The solution φ can be expressed in terms of a Green's function,

$$\varphi(\zeta) = \int G(z, \zeta; k, l)F(z)dz, \quad \text{(A2)}$$

where

$$\frac{\partial}{\partial z}\left(\epsilon\frac{\partial G}{\partial z}\right) + \left(\frac{\partial\bar{q}_0/\partial y}{\bar{u}_0 - c_0} - k^2 - l^2\right)G = \delta(z - \zeta). \quad \text{(A3)}$$

But since

$$\frac{\partial}{\partial z}\left(\epsilon\frac{\partial\psi_0}{\partial z}\right) + \left(\frac{\partial\bar{q}_0/\partial y}{\bar{u}_0 - c_0} - k^2 - l^2\right)\psi_0 = -l^2\psi_0, \quad \text{(A4)}$$

we also have

$$\int G(z, \zeta; k, l)\psi_0(z)dz = -l^{-2}\psi_0(\zeta). \quad \text{(A5)}$$

Defining $H(z, \zeta; k, l)$ by

$$G = H - l^{-2}\gamma_2^{-1}\psi_0(\zeta)\psi_0(z), \quad \text{(A6)}$$

where $\gamma_2 = \int \psi_0^2 dz$ so that $\int H\psi_0 dz = 0$, we find that

$$\frac{\partial}{\partial z}\left(\epsilon\frac{\partial H}{\partial z}\right) + \left(\frac{\partial\bar{q}_0/\partial y}{\bar{u}_0 - c_0} - k^2 - l^2\right)H = \delta(z - \zeta) - \gamma_2^{-1}\psi_0(z)\psi_0(\zeta). \quad \text{(A7)}$$

As $l \to 0$, H approaches the generalized Green's function defined by Courant and Hilbert (1953) for differential operators possessing a zero eigenvalue. Therefore $H = O(l^0)$ as $l \to 0$. It follows that

$$\lim_{l\to 0} \varphi(\zeta; k, l) = -l^{-2}\gamma_2^{-1}\psi_0(\zeta)$$
$$\times \int \frac{\psi_0}{(\bar{u}_0 - c_0)}\left(\psi_0 \frac{\partial}{\partial z}\left(\epsilon \frac{\partial g}{\partial z}\right) - q_0 g\right)dz. \quad \text{(A8)}$$

From (3.10), we immediately obtain the formula (3.14) for $\lim_{l\to 0} m(k, l)$.

If k is now allowed to approach either the short or long-wave cutoff of the unperturbed problem, so that $c_{I,0} \downarrow 0$, one can show that (3.14) reduces to the $l \to 0$ limit of (3.17):

$$\left.\begin{aligned} \lim_{l\to 0} m(k, l) &= -2n_0\left[\alpha g + \frac{\partial}{\partial z}\left(\epsilon \frac{\partial g}{\partial z}\right)\right] \\ n_0 &= \frac{k\pi}{2} |\psi_0|^2 \left|\frac{\partial \bar{u}}{\partial z}\right|^{-1} \\ \alpha &= \frac{\partial^2 \bar{q}_0}{\partial y \partial z} \Big/ \frac{\partial \bar{u}_0}{\partial z} \end{aligned}\right\} \quad \text{(A9)}$$

with all quantities evaluated at z_0, the steering level of the unperturbed mode as $c_{I,0} \downarrow 0$.

To see this, note first that we are free to choose $\mathrm{Im}(\psi_0) \to 0$ as $c_{I,0} \downarrow 0$, so that $\gamma_1 \to \gamma_2$. Also,

$$\lim_{c_{I,0}\downarrow 0} \mathrm{Im} \int \frac{f(z, c_{I,0})}{(\bar{u}_0 - c_0)}\, dz = \frac{\pi}{\left|\dfrac{\partial \bar{u}}{\partial z}\right|_{z_0}} f(z_0, 0), \quad \text{(A10)}$$

if f is real. But

$$q_0 \to -\frac{\psi_0 \partial \bar{q}_0/\partial y}{\bar{u}_0 - c_{0,R}} \text{ as } c_{0,I} \downarrow 0, \quad \text{(A11)}$$

$$\lim_{z\to z_0} \frac{\partial \bar{q}_0/\partial y}{\bar{u}_0 - c_{0,R}} = \frac{\partial^2 \bar{q}_0/\partial y \partial z}{\partial \bar{u}_0/\partial z}, \quad \text{(A12)}$$

using L'Hospital's rule. Combining (3.14) with (A10)–(A12), we immediately obtain (A9) once again.

REFERENCES

Bretherton, F. P., 1966: Critical layer instability in baroclinic flows. *Quart. J. Roy. Meteor. Soc.*, **92**, 325–334.

Brown, J. A., Jr., 1969: A numerical investigation of hydrodynamic instability and energy conversions in the quasi-geostrophic atmosphere: Part I. *J. Atmos. Sci.*, **26**, 352–365.

Charney, J. G., and M. F. Stern, 1962: On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.*, **19**, 159–172.

Conrath, B. J., P. J. Gierasch and N. Nath, 1981: Stability of zonal flows on Jupiter. *Icarus*, **48**, 256–282.

Courant, R., and D. Hilbert, 1953: *Methods of Mathematical Physics*, Vol. I. Interscience, 560 pp.

Drazin, P. G., and L. N. Howard, 1966: Hydrodynamic stability of parallel flow of inviscid fluid. *Advances in Applied Mathematics*, Vol. 9, Academic Press, 1–89.

Gall, R., 1976: Structural changes of growing baroclinic waves. *J. Atmos. Sci.*, **33**, 374–390.

Gent, P., 1975: Baroclinic instability of a slowly varying zonal flow. Part 2. *J. Atmos. Sci.*, **32**, 2094–2102.

Green, J. S. A., 1970: Transfer properties of large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **96**, 157–185.

Hart, J. E., 1974: On the mixed stability problem for quasi-geostrophic ocean currents. *J. Phys. Oceanogr.*, **4**, 349–356.

Held, I. M., 1975: Momentum transport by quasi-geostrophic eddies. *J. Atmos. Sci.*, **33**, 1494–1497.

Holland, W. R., and D. B. Haidvogel, 1980: A parameter study of the mixed instability of idealized ocean currents. *Dyn. Atmos. Oceans*, **4**, 185–215.

Hoskins, B. J., and N. V. West, 1979: Baroclinic waves and frontogenesis. Part II: Uniform potential vorticity jet flows—cold and warm fronts. *J. Atmos. Sci.*, **36**, 1663–1680.

Killworth, P. D., 1981: Eddy fluxes and mean flow tendencies in open-ocean baroclinic instability. *Dyn. Atmos. Oceans*, **5**, 175–186.

Lau, N.-C., 1978: On the three-dimensional structure of the observed transient eddy statistics of the Northern Hemisphere wintertime circulation. *J. Atmos. Sci.*, **35**, 1900–1923.

McIntyre, M. E., 1970: On the non-separable baroclinic parallel flow instability problem. *J. Fluid Mech.*, **40**, 273–306.

Pedlosky, J., 1964: The stability of currents in the atmosphere and the ocean: Part II. *J. Atmos. Sci.*, **21**, 342–353.

Phillips, N. A., 1951: A simple three-dimensional model for the study of large-scale extratropical flow patterns. *J. Meteor.*, **8**, 381–394.

Simmons, A. J., 1974: The meridional scale of baroclinic waves. *J. Atmos. Sci.*, **31**, 1515–1525.

——, and B. J. Hoskins, 1978: The life cycles of some nonlinear baroclinic waves. *J. Atmos. Sci.*, **35**, 414–432.

Stone, P. H., 1969: The meridional structure of baroclinic waves. *J. Atmos. Sci.*, **26**, 376–389.