

Potential Vorticity Thickness Fluxes and Wave–Mean Flow Interaction

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ABSTRACT

The use of eddy flux of thickness between density surfaces has become a familiar starting point in oceanographic studies of adiabatic eddy effects on the mean density distribution. In this study, a dynamical analogy with the density thickness flux approach is explored to reexamine the theory of nonzonal wave–mean flow interaction in two-dimensional horizontal flows. By analogy with the density thickness flux, the flux of thickness between potential vorticity (PV) surfaces is used as a starting point for a residual circulation formulation for nonzonal mean flows. Mean equations for barotropic PV dynamics are derived in which a modified mean velocity with an eddy-induced component advects a modified mean PV that also has an eddy-induced component. For small-amplitude eddies, the results are analogous to recent results of McDougall and McIntosh derived for stratified flow.

The dynamical implications of this approach are then examined. The modified mean PV equation provides a decomposition of the eddy forcing of the mean flow into contributions from wave transience, wave dissipation, and wave-induced mass redistribution between PV contours. If the mean flow is along the mean PV contours, the contribution from wave-induced mass redistribution is “workless” in Plumb’s sense that it is equivalent to an eddy-induced stress that is perpendicular to the mean flow. This contribution is also associated with the convergence along the mean streamlines of a modified PV flux that is equal to the difference between the PV flux and the rotational PV flux term identified by Illari and Marshall. The cross-stream component of the modified PV flux is related to wave transience and dissipation.

1. Introduction

Much of what we understand about the role of large-scale extratropical eddies in the atmospheric general circulation is based on the theory of the interaction of these eddies with zonally symmetric mean flows. The various parts of the zonal wave–mean flow interaction theory—for example, the “nonacceleration” theorem, the transformed Eulerian mean (TEM) “residual” circulation framework, and the Eliassen–Palm (E–P) flux diagnostics—have provided us with a useful and coherent framework for thinking about eddy-driven processes from a variety of perspectives [see Andrews et al. (1987) for a review of the theory, applications, and references].

Extending the zonal theory to a theory of the interaction of eddies with nonzonal mean flows has proven to be complicated. For example, Plumb’s (1990) nonacceleration theorem for nonzonal time mean flows (see also Andrews 1990) states that eddy-driven changes to the mean flow can be brought about by wave transience and dissipation, as in the zonal theory. But in addition,

alongstream variations in certain eddy statistics also bring about mean flow changes, even if the eddies are steady in time and conservative. Other examples of approaches to a nonzonal theory are the generalizations of the E–P flux diagnostics to the so-called **E**-vector flux (Hoskins et al. 1983) and to Plumb’s (1986) generalized wave activity. These flux diagnostics highlight important effects, such as the relevance of meridional–longitudinal eddy anisotropy in driving mean flow changes, but have not been satisfactorily connected to a nonacceleration theorem or to a “wave activity” conservation law, as in the zonal theory.

To make progress toward a more coherent wave–mean flow interaction theory for zonally asymmetric flows, we propose to explore an analogy to a common approach in studies of oceanic mesoscale eddies and in subgrid-scale eddy flux parameterizations for oceanic general circulation models. These studies (e.g., Rhines and Holland 1979; Gent and McWilliams 1990; Gent et al. 1995; Treguier et al. 1997) take the flux of mass between isopycnal surfaces as a starting point for understanding the eddy driving of mean flows. If we define the thickness $h = \partial z / \partial b$, with h positive and finite, where b is the buoyancy of a Boussinesq fluid and $z(x, y, b, t)$ is the height of a buoyancy surface, then the continuity equation and the thermodynamic equation imply that the mean thickness equation in an adiabatic region is

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$$\left. \frac{\partial \bar{h}^b}{\partial t} \right|_b + \nabla_b \cdot (\mathbf{V}^{*b} \bar{h}^b) = 0. \quad (1.1)$$

Here \bar{A}^b represents an average of A along instantaneous buoyancy surfaces, the subscript b indicates a derivative at fixed buoyancy, and

$$\mathbf{V}^{*b} = \bar{\mathbf{V}}^b + \frac{(\overline{\mathbf{V}'^b h'^b})}{\bar{h}^b} \quad (1.2)$$

is an effective mean horizontal velocity that consists of a mean part, $\bar{\mathbf{V}}^b$, and an eddy part, $(\overline{\mathbf{V}'^b h'^b})/\bar{h}^b$. In (1.2), the notation “ $'b$ ” refers to a disturbance quantity that is evaluated at fixed b . The quantity $\mathbf{V}^{*b} \bar{h}^b$ is the mean thickness flux, and eddy flux parameterization schemes consist, in part at least, of a theory for the eddy contribution to this thickness flux.

In the mean thickness equation (1.1) the eddies, apart from diabatic effects, redistribute mass between buoyancy surfaces. Diabatic effects would appear as mass sources and sinks. The mean thickness equation cleanly sorts out adiabatic from diabatic eddy processes. This contrasts with the mean buoyancy equation derived in geometric z coordinates, which includes the averaged eddy buoyancy flux as an apparent diapycnal term, and in which the mean flow can have a component across isopycnal surfaces, even for adiabatic conditions.

Given the advantages of using the buoyancy–thickness flux approach for understanding eddy-induced circulations in stratified flows, we propose here to turn the approach on its side and use the flux of mass or thickness between potential vorticity (PV) surfaces in a two-dimensional horizontal (barotropic) model as a starting point for understanding eddy-induced circulations in zonally asymmetric flows. We define the PV thickness $h = \partial y / \partial q$, with h positive and finite, where q is the PV and $y(x, q, t)$ is the meridional position of the PV contour. Then, for conservative nondivergent barotropic flow we find, analogously to the mean buoyancy thickness equation (1.1) (see section 2),

$$\left. \frac{\partial \bar{h}^q}{\partial t} \right|_q + \left. \frac{\partial}{\partial x} (u^{*q} \bar{h}^q) \right|_q = 0, \quad (1.3)$$

where

$$u^{*q} = \bar{u}^q + \frac{(\overline{u'^q h'^q})}{\bar{h}^q}. \quad (1.4)$$

Similarly to the case for density thickness, the induced eddy velocity in (1.3) has the effect of conservatively redistributing mass between PV surfaces, and nonconservative processes act as mass sources or sinks. For reversible PV dynamics, there is no evident constraint that prevents the simplest linear waves from redistributing the mass between PV contours through a correlation between h'^q and u'^q . This is the key to the extra complexity in wave–mean flow interaction theory on zonally asymmetric flows. For example, we could

imagine an eddy-induced circulation that advects mean thickness anomalies in the alongstream direction, or an eddy-induced thickness anomaly that is advected by the mean circulation.

In section 2 of this paper, we explore the overall similarity between the mean PV thickness equation (1.3) and the mean buoyancy thickness equation (1.1) to derive a residual circulation formulation of mean barotropic PV dynamics that is essentially a special case of the three-dimensional temporal-residual mean (TRM) circulation derived by McDougall and McIntosh (1996) and McDougall and McIntosh (1999, manuscript submitted to *J. Phys. Oceanogr.*). Hereafter, these two papers will be referred to as MM. We discuss the PV coordinate mean PV thickness equation (1.3) and point out how this equation can be written in geometric coordinates as an equation in which a “modified mean PV” is advected by the effective transport velocity. Details of the derivations in section 2 are presented in appendixes A and B. In appendix C, a similar modified mean PV equation is derived by manipulation of the mean PV and eddy enstrophy equations. This derivation is similar to MM.

In section 3, we examine the dynamical implications of the PV thickness flux approach and the modified mean PV equation. We decompose the modified mean PV equation, for small-amplitude eddies, into terms associated with wave dissipation, wave transience, and wave-induced mass-redistribution effects of the kind described above. This decomposition leads to interesting points of contact between these results and the work of Plumb (1990) and Illari and Marshall (1983).

The results are summarized in section 4, and applications, limitations, and generalizations of the work are discussed in section 5.

2. PV thickness and modified mean PV equations

The dynamical equation for the barotropic PV, q , is

$$\frac{Dq}{Dt} = \left. \frac{\partial q}{\partial t} \right|_y + \mathbf{u} \cdot \nabla_y q = \left. \frac{\partial q}{\partial t} \right|_y + \partial_{xy}(\psi, q) = S, \quad (2.1)$$

where D/Dt is the material derivative, $\mathbf{u} = (u, v) = (-\partial_y \psi, \partial_x \psi|_y)$ is the nondivergent horizontal velocity with streamfunction ψ , and $\partial_{xy}(f, g) = \partial_x f|_y \partial_y g - \partial_y f \partial_x g|_y$ is the Jacobian. The right-hand side of the PV equation (2.1), S , represents PV sources or sinks. We note that throughout this section q can also represent any materially conserved tracer. The subscript y on the derivatives serves to distinguish the derivative at a fixed point in space from the derivative at fixed PV in the following.

In Eq. (2.1), the coordinates x and y can represent the Cartesian zonal and meridional coordinates, but, except as noted below, all of the derivations that follow are valid if x and y are generalized to any orthogonal curvilinear time-independent coordinate system (ξ, η) with

$\partial_{xy}(\xi, \eta) \equiv 1$ everywhere. In this case, $\partial_r f|_{x,y} = \partial_r f|_{\xi,\eta}$ and $\partial_{xy}(f, g) = \partial_{\xi\eta}(f, g)$. In particular, one can choose ξ to be tangential and η to be normal to mean streamlines or to mean PV contours.

We begin by presenting a mean thickness equation for the PV. We will assume that the PV thickness,

$$h = \left(\frac{\partial q}{\partial y}\right)^{-1} = \frac{\partial y}{\partial q}, \tag{2.2}$$

is nonzero and finite in some region of interest. It is usually understood that the meridional PV gradient and thus h are positive. On the one hand, this assumption is not as restrictive as it seems since we are free to choose the x coordinate to be along the mean PV contours. On the other hand, we have excluded situations in which the contours roll up or shed filaments. For the narrow purpose of this paper, this is desirable since we would like to isolate and understand the wave-mean flow interaction that persists even in the absence of wave breaking and the associated PV mixing. We return to this point in section 5.

The thickness h is the Jacobian of the transformation from y to q coordinates. The requirement that h is locally nonzero and finite allows us to locally transform the dynamics to q coordinates. Starting with the barotropic PV equation (2.1), we show in appendix A how to obtain, using manipulations analogous to those used to obtain the buoyancy thickness equation, a prognostic equation for the PV thickness:

$$\left.\frac{\partial h}{\partial t}\right|_q + \left.\frac{\partial(uh)}{\partial x}\right|_q = -\frac{\partial(Sh)}{\partial q}. \tag{2.3}$$

Equation (2.3) recasts the dynamics in terms of a thickness or mass flux between PV contours.

The fixed q mean of the thickness equation (2.3) is, without approximation,

$$\begin{aligned} \left.\frac{\partial \bar{h}^q}{\partial t}\right|_q + \left.\frac{\partial}{\partial x}[\bar{u}^q \bar{h}^q + \overline{(u'^q h'^q)^q}]\right|_q \\ = -\frac{\partial}{\partial q}[\bar{S}^q \bar{h}^q + \overline{(S'^q h'^q)^q}]. \end{aligned} \tag{2.4}$$

As in section 1, \bar{A}^q denotes the average and A'^q the perturbation A at fixed q . (The particular averaging operation will be left unspecified; the ‘‘eddies’’ are then departures from whatever average is chosen.) The mean thickness in (2.4), $\bar{h}^q = \partial \bar{y}^q / \partial q$, is the q derivative of the mean contour position \bar{y}^q .

The combined mean and eddy terms in the x flux of mass in (2.4) may be expressed in terms of an effective transport velocity, u^{*q} , defined previously by (1.4). (We use the convention that the superscript ‘‘* q ’’ denotes a mean quantity defined in q coordinates that includes mean and eddy contributions.) Similarly, the eddy and mean terms in the thickness-weighted nonconservative term may be combined into an effective nonconservative term, S^{*q} , defined by

$$S^{*q} = \bar{S}^q + \frac{\overline{(S'^q h'^q)^q}}{\bar{h}^q}. \tag{2.5}$$

Then the mean PV thickness equation (2.4) may be written

$$\left.\frac{\partial \bar{h}^q}{\partial t}\right|_q + \left.\frac{\partial}{\partial x}(u^{*q} \bar{h}^q)\right|_q = -\frac{\partial}{\partial q}(S^{*q} \bar{h}^q). \tag{2.6}$$

Equation (2.6) reduces to (1.3) when $S = 0$.

The mean PV thickness equation (2.6) provides an intuitively clear separation between conservative and nonconservative eddy-induced effects. If the thickness perturbations h'^q and the velocity perturbations u'^q are correlated, then the eddies simply redistribute mass between PV contours. If, on the other hand, the thickness perturbations are correlated with the source-term perturbations, S'^q , then the eddies induce effective mass sources or sinks between the contours.

It is often desirable to express the PV coordinate mean equation (2.6) in geometric coordinates. To derive the geometric-coordinate mean equation, we transform the mean thickness equation (2.6) from q coordinates to \bar{y}^q coordinates, where we recall that \bar{y}^q is the mean meridional coordinate of the PV contour with value q . DeSzoek and Bennett (1993) consider the analogous transformation from density to mean height coordinates. The mean thickness $\bar{h}^q = \partial_q[\bar{y}^q(q)]$ is the Jacobian of the transformation and is assumed to be sign definite, which is approximately equivalent to assuming that the mean PV gradient is sign definite. [Note that functions such as $\bar{y}^q(q)$ are in general x and t dependent, but here and below, any such dependence is only made explicit as necessary.]

We may define a mean streamfunction,

$$\psi^{*q}[\bar{y}^q(q)] = -\int^{\bar{y}^q(q)} u^{*q}(\hat{y}) d\hat{y}, \tag{2.7}$$

so that

$$(u^{*q}, v^{*q}) = \left(-\frac{\partial \psi^{*q}}{\partial \bar{y}^q}, \frac{\partial \psi^{*q}}{\partial x}\right)_{\bar{y}^q}, \tag{2.8}$$

where we have defined the meridional component of the effective transport velocity, v^{*q} , in terms of the streamfunction ψ^{*q} . With (2.7), we obtain, by a coordinate transformation of the mean thickness equation (2.6), the following mean equation for q :

$$\left.\frac{\partial q}{\partial t}\right|_{\bar{y}^q} + \partial_{x\bar{y}^q}(\psi^{*q}, q) = S^{*q}. \tag{2.9}$$

[The derivation uses similar manipulations to those used in appendix A to derive the thickness equation (2.3) from the PV equation (2.1).] This is a mean equation in the sense that q is now independent of any ‘‘fast’’ coordinate (for example, a fast timescale or the wave phase), just as is $\bar{y}^q(q)$ in the mean thickness equation (2.6).

We now modify the notation of Eq. (2.9) to write the equation in y coordinates rather than in \bar{y}^q coordinates. We define a particular value of PV, denoted q^{*q} , whose mean contour position is the meridional coordinate y . That is, we define

$$q^{*q} \text{ such that } \bar{y}^q = y \text{ when } q = q^{*q}. \quad (2.10)$$

We will call q^{*q} the modified mean PV and note that it should be distinguished from the geometric-coordinate mean PV, \bar{q}^y . An analogous quantity for density in stratified flow has been defined by deSzoeke and Bennett (1993) and MM. Using (2.10), (2.9) becomes

$$\left. \frac{\partial q^{*q}}{\partial t} \right|_y + \partial_{xy}(\psi^{*q}, q^{*q}) = S^{*q}. \quad (2.11)$$

The modified mean PV equation (2.11) expresses the advection of the modified mean PV by the effective transport velocity, which has streamfunction ψ^{*q} . Equation (2.11) has been derived without approximation from the mean thickness equation and so essentially follows from mass conservation.

In appendix B, we use Taylor series expansions to derive expressions for q^{*q} , ψ^{*q} , and S^{*q} for small-amplitude perturbations to the mean. To second order in the eddy amplitude, the modified mean PV is

$$q^{*q} = \bar{q}^y + \bar{q}_{\text{eddy}}, \quad \text{where } \bar{q}_{\text{eddy}} = -\frac{\partial}{\partial y} \left[\frac{1}{2} \frac{\overline{(q')^2}}{\partial_y \bar{q}} \right], \quad (2.12)$$

the effective transport streamfunction is

$$\begin{aligned} \psi^{*q} &= \bar{\psi}^y + \bar{\psi}_{\text{eddy}}, \quad \text{where} \\ \bar{\psi}_{\text{eddy}} &= \frac{\overline{u'^y q'}}{\partial_y \bar{q}} - \frac{1}{2} \frac{\partial \bar{u}}{\partial y} \frac{\overline{(q')^2}}{(\partial_y \bar{q})^2}, \end{aligned} \quad (2.13)$$

and the effective nonconservative term is

$$\begin{aligned} S^{*q} &= \bar{S}^y + \bar{S}_{\text{eddy}}, \quad \text{where} \\ \bar{S}_{\text{eddy}} &= \frac{\partial}{\partial y} \left[-\frac{\overline{(S'^y q')}}{\partial_y \bar{q}} + \frac{1}{2} \frac{\partial \bar{S}}{\partial y} \frac{\overline{(q')^2}}{(\partial_y \bar{q})^2} \right]. \end{aligned} \quad (2.14)$$

In Eqs. (2.12)–(2.14), the subscript “eddy” indicates a second-order eddy contribution, and the superscript y and q notation is used only as necessary to retain accuracy to second order. [In particular, the difference between averages in q and y coordinates is second order in eddy amplitude (see appendix B), so it is not necessary to use the notation for eddy terms or for mean terms multiplying eddy terms.] These expressions show explicitly how the modified mean PV, the effective transport streamfunction, and the effective nonconservative term all differ from their geometric-coordinate mean counterparts by second-order quantities.

For small-amplitude eddies, it is possible to derive equation (2.11) with (2.12)–(2.14) directly from the mean PV equation

$$\left. \frac{\partial \bar{q}^y}{\partial t} \right|_y + \bar{\mathbf{u}}^y \cdot \nabla \bar{q}^y + \nabla \cdot \overline{(\mathbf{u}'^y q')}^y = \bar{S}^y \quad (2.15)$$

and the eddy enstrophy equation

$$\begin{aligned} \frac{\overline{D}^y}{Dt} \left[\frac{1}{2} \overline{(q')^2}^y \right] + \overline{(\mathbf{u}'^y q')}^y \cdot \nabla \bar{q}^y + \left[\overline{\mathbf{u}'^y \cdot \nabla \left(\frac{1}{2} (q')^2 \right)}^y \right] \\ = \overline{(S'^y q')}^y, \end{aligned} \quad (2.16)$$

where $\overline{D}^y A/Dt = \partial_t A|_y + \bar{\mathbf{u}}^y \cdot \nabla_y A$. The derivation, shown in appendix C, is similar to that of MM's TRM equations. It is more algebraically difficult than the development above and apparently does not generalize to finite-amplitude eddies; for finite-amplitude eddies, we obtain an expression similar to (2.11) with additional order-amplitude-cubed terms on the right-hand side. [See Eq. (C10).]

3. Dynamics of the modified mean PV equation

So far, the development has been kinematic and we have made little direct contact with a specific dynamical model. In fact, as was mentioned in section 2, the derivation of the modified mean PV equation (2.11) would be equally valid for any tracer q advected by a horizontal nondivergent velocity in the presence of some nonconservative term S , provided the tracer thickness $\partial_y q$ was sufficiently well behaved that q would be a reasonable meridional coordinate. We will now explore some of the dynamical implications of this thickness flux framework, with a particular focus on wave–mean flow interaction theory. The results in this section are restricted to small-amplitude eddies.

- a. The modified mean PV equation provides a decomposition of wave driving into parts due to wave transience, wave dissipation, and wave-induced mass redistribution*

The mean PV equation (2.15) can be rewritten

$$\frac{\partial \bar{q}^y}{\partial t} + \bar{\mathbf{u}}^y \cdot \nabla \bar{q}^y - \bar{S}^y = -\nabla \cdot \overline{(\mathbf{u}'^y q')}^y. \quad (3.1)$$

(We assume small-amplitude eddies, and all x and t derivatives are understood to be taken at fixed y unless otherwise noted.) In (3.1), the mean dynamics can be thought of as being forced by the PV flux divergence $\nabla \cdot \overline{(\mathbf{u}'^y q')}^y$. This interpretation of the eddy forcing of the mean dynamics is a common starting point for diagnostic studies of the role of eddy forcing of zonally asymmetric flows (e.g., Hoskins et al. 1983).

On the other hand, from the modified mean PV equation (2.11) with the small-amplitude expressions (2.12)–(2.14), we see that, to second order in eddy amplitude,

$$\frac{\partial \bar{q}^y}{\partial t} + \bar{\mathbf{u}}^y \cdot \nabla \bar{q}^y - \bar{S}^y = -\frac{\partial \bar{q}_{\text{eddy}}}{\partial t} - \nabla \cdot \mathbf{P} + \bar{S}_{\text{eddy}}, \quad (3.2)$$

where

$$\begin{aligned} \nabla \cdot \mathbf{P} &\equiv \partial_{xy}(\bar{\psi}_{\text{eddy}}, \bar{q}) + \partial_{xy}(\bar{\psi}, \bar{q}_{\text{eddy}}) \\ &= \bar{\mathbf{u}}_{\text{eddy}} \cdot \nabla \bar{q} + \bar{\mathbf{u}} \cdot \nabla \bar{q}_{\text{eddy}}, \end{aligned} \quad (3.3)$$

and where we can choose

$$\mathbf{P} = \bar{\mathbf{u}}_{\text{eddy}} \bar{q} + \bar{\mathbf{u}} \bar{q}_{\text{eddy}}. \quad (3.4)$$

In (3.3)–(3.4), $\bar{\mathbf{u}}_{\text{eddy}} = \hat{\mathbf{k}} \times \nabla \bar{\psi}_{\text{eddy}}$. In (3.2), the mean dynamics can be thought of as being forced by a combination of wave transience ($\partial \bar{q}_{\text{eddy}}/\partial t$), wave dissipation (\bar{S}_{eddy}), and a term that remains even when the eddies are steady and conservative ($\nabla \cdot \mathbf{P}$). This alternative interpretation is reminiscent of zonal wave–mean flow interaction theory.

An important difference from the zonal theory is the presence of the term $\nabla \cdot \mathbf{P}$ in (3.2). The term $\nabla \cdot \mathbf{P}$ consists of a contribution [$\bar{\mathbf{u}}_{\text{eddy}} \cdot \nabla \bar{q}$ in (3.3)] that represents advection by a wave-induced velocity of the mean PV and a contribution [$\bar{\mathbf{u}} \cdot \nabla \bar{q}_{\text{eddy}}$ in (3.3)] that represents advection by the mean velocity of a wave-induced PV contribution. We have, from (3.1) and (3.2),

$$\begin{aligned} \nabla \cdot (\overline{\mathbf{u}'^y q'}) &= \frac{\partial \bar{q}_{\text{eddy}}}{\partial t} + \nabla \cdot \mathbf{P} - \bar{S}_{\text{eddy}}, \\ &\text{for small-amplitude eddies,} \end{aligned} \quad (3.5)$$

and

$$\nabla \cdot (\overline{\mathbf{u}'^y q'}) = \nabla \cdot \mathbf{P}, \quad \text{for steady and conservative small-amplitude eddies.} \quad (3.6)$$

By definition of the transformation to mean-height coordinates discussed in section 2, we have

$$\partial_x(\overline{u'^q h^q})|_{q=q^*q} = -\frac{\partial}{\partial q^*q} [\bar{h}^q \partial_{xy}(\psi'^q, q'^q)]. \quad (3.7)$$

This shows that the thickness flux in the mean thickness equation (2.6) and the advection term in the modified mean PV equation (2.11) are directly related. Using the notation of this section, (3.7) becomes

$$\begin{aligned} \partial_x [\overline{u^q(q^*q) h^q(q^*q)} + \overline{(u'^q h'^q)}] |_{q=q^*q} \\ = -\frac{\partial}{\partial q^*q} [\bar{h}^q(q^*q) [\partial_{xy}(\bar{\psi}^y, \bar{q}^y) + \nabla \cdot \mathbf{P}]]. \end{aligned} \quad (3.8)$$

Equation (3.8) shows that the total (i.e., mean plus eddy) thickness flux is directly related to the total modified mean PV advection. However, there is no evident direct relation between the eddy terms on each side of (3.8); part of the difficulty in determining such a relation results from the fact that q and y coordinate means themselves differ by second-order eddy quantities. It is nevertheless clear from the equation that, as seen from a y -coordinate perspective, $\nabla \cdot \mathbf{P}$ is the eddy contribution

to the thickness advection. We therefore interpret $\nabla \cdot \mathbf{P}$ as a wave-induced mass redistribution term and, using (3.5)–(3.6), interpret the flux \mathbf{P} as the part of the PV flux $\overline{\mathbf{u}'^y q'}$ associated with wave-induced mass redistribution.

b. The P flux can be associated with a “workless” eddy stress

The mass-redistributing part of the eddy PV flux, \mathbf{P} , is related to a mechanical stress on the mean flow. The vector $-\hat{\mathbf{k}} \times \overline{\mathbf{u}'^y q'}$ is often thought of as a force on the $\bar{\mathbf{u}}^y$ component of the mean momentum equations (Plumb 1990; Hoskins 1983). So the mechanical stress $-\hat{\mathbf{k}} \times \mathbf{P}$ that is associated with the term $\bar{\mathbf{u}} \bar{q}_{\text{eddy}}$ in (3.4) is always perpendicular to the mean flow, and therefore “workless,” as Plumb (1990) puts it. In the special case that the mean flow is along the mean PV contours, the mean streamfunction is a function of \bar{q} . In this case, it is straightforward to show that

$$\nabla \cdot \mathbf{P} = (\hat{\mathbf{k}} \times \nabla B) \cdot \nabla \bar{q} = \nabla \cdot \left(-\bar{\mathbf{u}} \frac{d\bar{q}}{d\bar{\psi}} B \right), \quad (3.9)$$

where B is an averaged eddy quantity defined by

$$B = \frac{\overline{u'^y q'}}{\partial_y \bar{q}} + \frac{1}{\partial_y \bar{q}} \frac{\partial}{\partial y} \left[\frac{1}{2} \frac{d\bar{\psi}}{d\bar{q}} (q')^2 \right]. \quad (3.10)$$

From (3.9) we can choose

$$\mathbf{P} = -\bar{\mathbf{u}} \frac{d\bar{q}}{d\bar{\psi}} B, \quad (3.11)$$

and the corresponding force $-\hat{\mathbf{k}} \times \mathbf{P}$ is workless.

c. The P flux is related to Illari and Marshall's (1983) modified PV flux

Illari and Marshall's (1983) [see also Marshall and Shutts (1981)] analysis of the mean and mean variance equations for the PV and the buoyancy provides another interesting perspective on the role of \mathbf{P} in the mean dynamics. Given a $\bar{\psi}$ – \bar{q} relation, they identify a rotational contribution to the PV flux,

$$[\overline{(\mathbf{u}'^y q')}]_{\text{R}} = \hat{\mathbf{k}} \times \nabla \left[\frac{d\bar{\psi}}{d\bar{q}} \frac{1}{2} (q')^2 \right]. \quad (3.12)$$

This contribution, whose dot product with $\nabla \bar{q}$ cancels the advection of eddy enstrophy in the eddy enstrophy equation (2.16), can have a substantial up-PV-gradient component, and, as seen in an example Illari and Marshall describe, can dominate the total PV flux obtained from meteorological data. Illari and Marshall point out that it is the remaining flux $\overline{(\mathbf{u}'^y q')} - [(\mathbf{u}'^y q')]_{\text{R}}$ that is dynamically important. We will call $\overline{(\mathbf{u}'^y q')} - [(\mathbf{u}'^y q')]_{\text{R}}$ the “modified PV flux.”

With (3.12), the definition of B in (3.10) can be written

$$B = \frac{\overline{(u'yq')} - [(u'yq')]_R}{\partial_y \bar{q}}, \quad (3.13)$$

giving, from (3.11),

$$\mathbf{P} = \bar{\mathbf{u}} \left[\frac{\overline{(u'yq')} - [(u'yq')]_R}{\bar{u}} \right]. \quad (3.14)$$

This expression for \mathbf{P} can be simplified further if the x direction is interpreted as running along mean streamlines and the y direction is interpreted as running perpendicular to mean streamlines. This choice of coordinates has been discussed in section 2. We will now use the following notation:

$$\begin{aligned} x & \text{ is the mean alongstream coordinate,} \\ y & \text{ is the mean cross-stream coordinate,} \\ u = \mathbf{u} \cdot \hat{\mathbf{x}} & \text{ is the velocity component in the } x \text{ direction,} \\ & \text{and} \\ v = \mathbf{u} \cdot \hat{\mathbf{y}} & \text{ is the velocity component in the } y \text{ direction.} \end{aligned} \quad (3.15)$$

In (3.15), $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ are the unit vectors in the x and y directions. In this coordinate system, the mean velocity is $\bar{\mathbf{u}} = \bar{u}\hat{\mathbf{x}}$, and therefore, using the coordinates (3.15),

$$\begin{aligned} \mathbf{P} &= \left\{ \overline{(u'yq')} - [(u'yq')]_R \right\} \hat{\mathbf{x}} \quad \text{and} \\ \nabla \cdot \mathbf{P} &= \frac{\partial}{\partial x} \left\{ \overline{(u'yq')} - [(u'yq')]_R \right\}. \end{aligned} \quad (3.16)$$

We have shown that given a $\bar{\psi}-\bar{q}$ relation and small-amplitude eddies, the mass-redistributing part of the PV flux reduces to the alongstream component of the modified flux.

d. The modified cross-stream PV flux can be directly related to wave transience and wave dissipation

The reduction (3.16) of \mathbf{P} to an alongstream vector leads to a decoupling of the along- and cross-stream components of the PV flux. By definition of the rotational component of the PV flux (3.12),

$$\nabla \cdot \overline{(\mathbf{u}'yq')} = \nabla \cdot \left\{ \overline{(\mathbf{u}'yq')} - [(u'yq')]_R \right\}. \quad (3.17)$$

Combining the equation for the mean dynamics (3.2) with (3.16), we find that

$$\begin{aligned} \nabla \cdot \overline{(\mathbf{u}'yq')} &= \frac{\partial \bar{q}_{\text{eddy}}}{\partial t} - \bar{S}_{\text{eddy}} \\ &+ \frac{\partial}{\partial x} \left\{ \overline{(u'yq')} - [(u'yq')]_R \right\}_y. \end{aligned} \quad (3.18)$$

Then (3.17) and (3.18) together imply that

$$\frac{\partial}{\partial y} \left\{ \overline{(u'yq')} - [(u'yq')]_R \right\}_x = \frac{\partial \bar{q}_{\text{eddy}}}{\partial t} - \bar{S}_{\text{eddy}}. \quad (3.19)$$

Equation (3.19) shows that the cross-stream component of the modified PV flux is associated with wave transience and dissipation. [This result, which is implicit in

Marshall and Shutts (1981) and Illari and Marshall (1983), can be directly obtained from the eddy enstrophy equation (2.16) in the alongstream coordinates (3.15), in which $\partial_x \bar{q}^y = 0$.] The alongstream component, however, is unconstrained by (3.17) and (3.18). For steady conservative eddies, (3.19) indicates that $\overline{(v'yq')} = [(v'yq')]_R$ and from (3.17) we have

$$\begin{aligned} \nabla \cdot \overline{(\mathbf{u}'yq')} &= \frac{\partial}{\partial x} \left\{ \overline{(u'yq')} \right\} + \frac{\partial}{\partial y} \left\{ \overline{(v'yq')} \right\}_R, \\ &= \frac{\partial}{\partial x} \left\{ \overline{(u'yq')} - [(u'yq')]_R \right\}, \quad \text{and} \\ &= \nabla \cdot \mathbf{P} \quad (\text{for steady conservative eddies}). \end{aligned} \quad (3.20)$$

From the definition of \bar{q}_{eddy} , (2.12), and the definition of \bar{S}_{eddy} , (2.14), Eq. (3.19) may be integrated in y , leading to

$$\overline{(v'yq')} - \overline{(v'yq')}_R = -\frac{\partial}{\partial t} \left[\frac{1}{2} \frac{\overline{(q')^2}}{\partial_y \bar{q}} \right] + \frac{\overline{(S'yq')}}{\partial_y \bar{q}}. \quad (3.21)$$

[Here, we have used that fact that \bar{S} is a second-order quantity for a slowly varying mean flow with a $\bar{\psi}-\bar{q}$ relation from (3.1) or (3.2).] Using the definition of the rotational component of the PV flux (3.12), this may be rewritten

$$\frac{\partial A}{\partial t} + \overline{(v'yq')} - \frac{\partial}{\partial x} \left[\frac{d\bar{\psi}}{d\bar{q}} \frac{1}{2} \overline{(q')^2} \right] = \frac{\overline{(S'yq')}}{\partial_y \bar{q}}, \quad (3.22)$$

where

$$A = \frac{1}{2} \frac{\overline{(q')^2}}{\partial_y \bar{q}}. \quad (3.23)$$

For a zonal mean flow, if the averaging operation is taken to be a zonal average, then A is the classic Eliassen–Palm wave activity, and $\overline{(v'yq')}$ can be rewritten as the divergence of the Eliassen–Palm flux. In this case, (3.22) reduces to the zonal wave-activity conservation law that can be derived directly from the PV equation (2.1) [see Andrews et al. (1987) for such a derivation]. In particular, A can be shown to be globally conserved in the absence of wave dissipation. One is tempted to generalize this zonal wave-activity conservation law to nonzonal mean flows by writing $\overline{(v'yq')}$ as the divergence of a flux, but this cannot be done straightforwardly in the curvilinear coordinates given by (3.15). Therefore, despite appearances, (3.22) is not a wave-activity conservation law.

4. Summary

We have introduced the idea that the PV thickness flux framework, following an analogy with the buoyancy thickness flux framework, provides a natural way to consider eddy-driven effects in zonally asymmetric mean flows. In this framework, steady and conservative

eddies redistribute mass between PV contours, and non-conservative effects are naturally associated with a loss or gain of mass between PV contours.

We have derived mean thickness and modified mean PV equations that are valid for finite-amplitude eddies. The modified mean PV equation is essentially a special case of MM's modified mean buoyancy, or TRM, equation. Our direct coordinate transformation approach is somewhat simpler than theirs and is valid for finite-amplitude eddies. One also has the choice of working in a Cartesian x - y coordinate system or in curvilinear coordinates following the mean flow.

We have then explored some dynamical implications of this framework. The modified mean PV equation decomposes the wave driving of the mean flow into effects of wave transience, wave dissipation, and wave-induced mass redistribution between PV contours. The wave-induced mass-redistribution can be shown to be locally workless in Plumb's sense and involves downstream convergence of the modified part of the PV flux that has been identified by Illari and Marshall (1983). The cross-stream component of the modified PV flux is directly related to wave transience and dissipation. The relationship between the PV flux and the eddy terms in the modified mean PV equation reduces to the zonal wave-activity conservation law for a zonal mean flow.

5. Discussion

An important question is whether it is valid to assume, as we have from the start, that the PV thickness is sign definite (usually positive definite) for realistic geophysical flows. In the presence of irreversible PV mixing and homogenization associated, for example, with Rossby wave breaking and the enstrophy cascade of geostrophic turbulence, the meridional PV gradient can become small or switch sign and hence the PV thickness can become locally ill-defined.

One way to address this question would be to "coarse grain" the PV distribution by passing it through a low-pass filter in space and time. The coarse graining could remove irreversibly deformed PV contours and leave behind a relatively well-behaved smoothed PV distribution with a well-defined thickness. This procedure would help separate reversible from irreversible eddy processes, which are both potentially important in non-zonal wave-mean flow interaction. The fluctuations of the coarse-grained PV contours could still produce conservative redistributions of mass as outlined in this paper, while the mass fluxes associated with the filtered-out features could play the role of the source term S in the PV equation (2.1).

Unfortunately, we know of few good model examples, besides Plumb's (1990) analysis of mean flow interaction effects for a wave train propagating through a stationary Rossby wave, to illustrate these ideas. A promising set of model problems involves waves propagating through a nonzonal PV distribution consisting

of a few PV contours separating regions of uniform PV, as analyzed by Swanson et al. (1997).

Another approach is to apply these ideas to analyze eddy-induced mean flow effects in an appropriate linear model. Recent work has shown that linear stochastic models (e.g., Delsole and Farrell 1995; Whitaker and Sardeshmukh 1998; Zhang and Held 1999) are able to reproduce important features of atmospheric eddy statistics in models and in GCMs, including the sensitivity of these statistics to changes in the mean flow. It should be fruitful to analyze the mean-flow tendencies predicted by linear stochastic models from this thickness flux perspective.

More ambitiously, one could try to analyze atmospheric observations from this perspective. This would amount to diagnoses of the mass budget of tubes bounded above and below by coarse-grained isentropic surfaces and to the north and south by coarse-grained isovortical surfaces. In this context, we note that the generalization of these barotropic results to three-dimensional quasigeostrophic (QG) dynamics is straightforward. If, instead of representing the barotropic PV, q is taken to represent the three-dimensional QG PV, then the present results carry through for QG dynamics by simply taking the streamfunction and the PV to be functions of x , y , and z , the QG dynamical equation being of the same form as the barotropic vorticity equation (2.1). One can apply the same analysis to the surface-temperature equation as well. Such an observational analysis would be of interest to the extent that the coarse graining resulted in a meaningful separation between reversible PV thickness redistribution and irreversible mixing.

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APPENDIX A

Details of Derivation of Eq. (2.3)

Assuming that h in (2.2) is positive definite, we may transform the PV equation (2.1) into q coordinates as follows:

$$\left. \frac{\partial y}{\partial t} \right|_q + u \left. \frac{\partial y}{\partial x} \right|_q - v = \left. \frac{\partial y}{\partial t} \right|_a - \left. \frac{\partial \psi}{\partial x} \right|_q = -Sh, \quad (\text{A1})$$

which is an equation for the contour position $y(x, q, t)$. To derive the contour equation (A1), we have used the following:

$$\left. \frac{\partial q}{\partial t} \right|_y = -\frac{1}{h} \left. \frac{\partial y}{\partial t} \right|_q, \quad \left. \frac{\partial q}{\partial x} \right|_y = -\frac{1}{h} \left. \frac{\partial y}{\partial x} \right|_q \quad (\text{A2})$$

and

$$u \left. \frac{\partial y}{\partial x} \right|_q - v = -\frac{\partial \psi}{\partial y} \left. \frac{\partial y}{\partial x} \right|_q - \left. \frac{\partial \psi}{\partial x} \right|_y = -\left. \frac{\partial \psi}{\partial x} \right|_q. \quad (\text{A3})$$

Alternatively, we may obtain (A1) directly by using $Dy/Dt = v$, with the material derivative expressed in q coordinates. Taking the q derivative of the contour equation (A1) gives

$$\left. \frac{\partial h}{\partial t} \right|_q - \left. \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \right) \right|_q = -\frac{\partial(Sh)}{\partial q}, \quad (\text{A4})$$

and Eq. (2.3) follows using

$$\frac{\partial \psi}{\partial q} = \frac{\partial \psi}{\partial y} \frac{\partial y}{\partial q} = -uh. \quad (\text{A5})$$

APPENDIX B

Some Results for Small-Amplitude Eddies

Consider a PV distribution, $q = q(y, \tau)$, that is a function of the independent coordinates y and τ . Here y is the meridional coordinate and τ is a fast coordinate in the sense that q 's dependence on τ may be averaged out with a suitable averaging operation. For example, τ could represent time, a spatial coordinate associated with short-scale variations, the phase of a wave, or an index indicating a particular member of an ensemble of realizations. We denote averaging over the τ coordinate by an overbar, and we have in the notation of section 2,

$$\overline{q(y, \tau)^y} = \overline{q^y} = \overline{q^y}(y). \quad (\text{B1})$$

We may then define the disturbance PV $q'^y = q'(y, \tau)$ by

$$q(y, \tau) = \overline{q^y}(y) + q'(y, \tau), \quad \text{with} \quad \overline{q'(y, \tau)^y} = 0. \quad (\text{B2})$$

(Henceforth we will use the superscript y or q notation only where necessary to avoid ambiguity.) Even if the averaging operation, such as time filtering over a finite time interval, does not entirely eliminate the τ dependence, it is understood that $|\partial_\tau [\log(\overline{q^y})]| \ll |\partial_\tau [\log(q')]|$.

For illustration we consider a simple model PV distribution,

$$q(y, t) = \beta y(1 + \epsilon \sin t)^{1/2}, \quad \epsilon \ll 1, \quad (\text{B3})$$

where $\tau = t$ is time and β is a constant. This PV distribution represents an oscillation with order ϵ amplitude about a β -plane distribution of PV. From (B3), we have

$$\begin{aligned} \overline{q^y}(y) &= \overline{\beta y(1 + \epsilon \sin t)^{1/2}} = \beta y \overline{(1 + \epsilon \sin t)^{1/2}} \\ &\approx \beta y(1 - \epsilon^2/16), \end{aligned} \quad (\text{B4})$$

where in this case the overbar represents a time average over a time interval much greater than unity, and in the

approximation we have neglected terms of $O(\epsilon^3)$. Equation (B4) indicates that in this example the presence of waves of order ϵ amplitude gives rise to an order ϵ^2 northward shift in the mean PV distribution.

A similar expansion for y yields

$$y(q, \tau) = \overline{y^q}(q) + y'(q, \tau), \quad \text{with} \quad \overline{y'(q, \tau)^q} = 0. \quad (\text{B5})$$

In our model example,

$$y = \beta^{-1} q(1 + \epsilon \sin t)^{-1/2}, \quad \text{so that}$$

$$\overline{y^q} = \beta^{-1} q(1 + 3\epsilon^2/16). \quad (\text{B6})$$

We assume that the disturbances are small in the usual sense that, for example, $|q'|/|\overline{q^y}| \ll 1$. (In our model example, $|q'|/|\overline{q^y}| \sim \epsilon \ll 1$.) Then a Taylor series expansion yields, for q ,

$$\begin{aligned} q &= \overline{q^y}[\overline{y^q}(q) + y'(q, \tau)] + q'[\overline{y^q}(q) + y'(q, \tau), \tau] \\ &\approx \overline{q^y}[\overline{y^q}(q)] + q'[\overline{y^q}(q), \tau] + \frac{\partial \overline{q^y}[\overline{y^q}(q)]}{\partial \overline{y^q}} y'(q, \tau) \\ &\quad + \frac{\partial q'(\overline{y^q}, \tau)}{\partial \overline{y^q}} y'(q, \tau) \\ &\quad + \frac{1}{2} \frac{\partial^2 \overline{q^y}[\overline{y^q}(q)]}{\partial \overline{y^q}^2} [y'(q, \tau)]^2, \end{aligned} \quad (\text{B7})$$

where in the approximation terms of up to second order in disturbance amplitude are retained.

Any function $g[\overline{y^q}(q)]$ can be obtained explicitly from function $g(y)$ by substituting the function $\overline{y^q}(q)$ for the y dependence in $g(y)$. In our model example,

$$\begin{aligned} \overline{q^y}[\overline{y^q}(q)] &= \overline{q(1 + \epsilon \sin t)^{-1/2} (1 + \epsilon \sin t)^{1/2}} \\ &= q[1 + \epsilon^2/8 + O(\epsilon^3)]. \end{aligned} \quad (\text{B8})$$

We now show, by taking the fixed q mean of (B7), that $\overline{q^y}[\overline{y^q}(q)] - q$ is, in general, a second-order quantity as it is in (B8). The fixed q mean of the left-hand side of (B7) is simply q . The quantity $\overline{q^y}[\overline{y^q}(q)]$, a mean (i.e., τ independent) function, is also unchanged by the fixed q mean: $\{\overline{q^y}[\overline{y^q}(q)]\}^q = \overline{q^y}[\overline{y^q}(q)]$. [For example, (B8) is unchanged by a fixed q mean.] In general, the fixed q mean of a function of $\overline{y^q}(q)$ is the same as its fixed $\overline{y^q}(q)$ mean. Then the quantity $q'[\overline{y^q}(q), \tau]$, which is the perturbation to $\overline{q^y}[\overline{y^q}(q)]$, vanishes under a fixed q mean, because $\{q'[\overline{y^q}(q), \tau]\}^q = \{q'[\overline{y^q}(q), \tau]\}^{\overline{y^q}} = 0$, where the last step follows from (B2). [For example, for our model distribution (B3),

$$\begin{aligned} q'[\overline{y^q}(q), \tau] &= \overline{q(1 + \epsilon \sin t)^{-1/2}} \\ &\quad \times [(1 + \epsilon \sin t)^{1/2} - \overline{(1 + \epsilon \sin t)^{1/2}}], \end{aligned}$$

whose fixed q mean clearly vanishes.] However, it should be noted that in general $\overline{(q')^q} = \{q'[y(q, \tau), \tau]\}^q \neq 0$. [For example, for our model distribution, it can be shown that $\overline{(q')^q} = -q\epsilon^2/8$ to second order.] Since $\overline{y'(q, \tau)^q} = 0$ by definition, we have to second order that

$$q = \bar{q}^y(\bar{y}^q) + \frac{\partial \bar{q}'}{\partial y} y' + \frac{1}{2} \frac{\partial^2 \bar{q}}{\partial y^2} (y')^2, \quad (\text{B9})$$

which shows that $q - \bar{q}^y[\bar{y}^q(q)]$ is indeed a second-order quantity.

Using (B7)–(B9) we find the familiar result that to first order in eddy amplitude

$$q' = -\frac{\partial \bar{q}}{\partial y} y', \quad (\text{B10})$$

which can be used to show that

$$q = \bar{q}^y(\bar{y}^q) - \frac{\partial}{\partial y} \left[\frac{1}{2} \frac{(q')^2}{\partial_y \bar{q}} \right]. \quad (\text{B11})$$

[For our model distribution,

$$\bar{q}^y[\bar{y}^q(q)] = q(1 + \epsilon^2/8) \text{ and } -\frac{\partial}{\partial y} \left[\frac{1}{2} \frac{(q')^2}{\partial_y \bar{q}} \right] = -q\epsilon^2/8$$

to second order, and the sum of these two terms is q , which is consistent with (B11).]

Now recall the definition of q^{*q} in (2.10). Taking $q = q^{*q}$ in (B11), we obtain (2.12).

Generalizing to any function $u(y, \tau)$ or $u(q, \tau)$, we find to second order that

$$\begin{aligned} u &\approx \bar{u}^y[\bar{y}^q(q)] + u'^y[\bar{y}^q(q), \tau] \\ &+ \frac{\partial \bar{u}^y[\bar{y}^q(q)]}{\partial \bar{y}^q} y'(q, \tau) + \frac{\partial u'^y[\bar{y}^q(q), \tau]}{\partial \bar{y}^q} y'(q, \tau) \\ &+ \frac{1}{2} \frac{\partial^2 \bar{u}^y[\bar{y}^q(q)]}{\partial \bar{y}^{q2}} [y'(q, \tau)]^2, \end{aligned} \quad (\text{B12})$$

where we have used the superscript “‘y’” notation to specify that the u perturbation is evaluated in y coordinates, but we note that $u'^y[\bar{y}^q(q), \tau]$ is really a function of q and τ . Similarly to (B9), we find

$$\bar{u}^q(q) = \bar{u}^y(\bar{y}^q) + \frac{\partial \bar{u}'^y}{\partial y} y' + \frac{1}{2} \frac{\partial^2 \bar{u}}{\partial y^2} (y')^2. \quad (\text{B13})$$

Equations (B12)–(B13) imply the standard result that to first order

$$u'^q = u'^y + \frac{\partial \bar{u}}{\partial y} y', \quad (\text{B14})$$

which, when substituted into (B13), gives

$$\begin{aligned} \bar{u}^q(q) + \frac{\partial \bar{u}'^y}{\partial y} y' &= \bar{u}^y(\bar{y}^q) + \frac{\partial}{\partial y} \left[\frac{\partial \bar{u}'^y}{\partial y} y' + \frac{1}{2} \frac{\partial \bar{u}}{\partial y} (y')^2 \right], \\ &= \bar{u}^y(\bar{y}^q) + \frac{\partial}{\partial y} \left[-\frac{\bar{u}'^y q'}{\partial_y \bar{q}} + \frac{1}{2} \frac{\partial \bar{u}}{\partial y} \frac{(q')^2}{(\partial_y \bar{q})^2} \right]. \end{aligned} \quad (\text{B15})$$

To derive (B15), we have used (B10) and the fact that $\partial_q y = \partial_q \bar{y}^q$ to first order. For q^{*q} defined in (2.10) we then have

$$\begin{aligned} \bar{u}^q(q^{*q}) + \frac{\partial \bar{u}'^y}{\partial y} y' &= \bar{u}^y(y) + \frac{\partial}{\partial y} \left[-\frac{\bar{u}'^y q'}{\partial_y \bar{q}} + \frac{1}{2} \frac{\partial \bar{u}}{\partial y} \frac{(q')^2}{(\partial_y \bar{q})^2} \right], \end{aligned} \quad (\text{B16})$$

or in the notation of section 2

$$\begin{aligned} \bar{u}^q(q^{*q}) + \frac{\partial \bar{u}'^y}{\partial y} y' &= \bar{u}^y(y) + \frac{\partial}{\partial y} \left[-\frac{\bar{u}'^y q'}{\partial_y \bar{q}} + \frac{1}{2} \frac{\partial \bar{u}}{\partial y} \frac{(q')^2}{(\partial_y \bar{q})^2} \right]. \end{aligned} \quad (\text{B17})$$

Taking u in (B17) to be the zonal component of the velocity, we see that the left-hand side of (B17) is u^{*q} in (1.4) evaluated at q^{*q} , and is, by (2.7), equal to $-\partial_y \psi^{*q}$ evaluated at q^{*q} . Equation (2.13) follows, since the right-hand side of (B17) is $-\partial_y$ of the right-hand side of the first equality in (2.13). Substituting the nonconservative term S for u in (B17), we obtain (2.14).

Analogous results for density as a function of height are derived by MM.

APPENDIX C

Derivation of the Modified Mean PV Equation Based on the Mean PV and Eddy Enstrophy Equations

We follow MM, who have constructed a similar mean equation for the buoyancy starting with the mean z -coordinate buoyancy and buoyancy variance equations. The mean equation of MM is itself a generalization of the transformed Eulerian mean formulation of Andrews and McIntyre (1976) (see Andrews et al. 1987). For quasigeostrophic scaling and zonally averaged dynamics, Andrews and McIntyre (1976) point out that the eddy buoyancy flux term can be removed by combining the zonally averaged velocity (\bar{v}^z, \bar{w}^z) with appropriate eddy terms. Here, v and w are the meridional and vertical components of the velocity, and “ \bar{A}^z ” indicates a fixed z zonal mean of A . In particular, Andrews and McIntyre (1976) define a residual velocity (v^{*z}, w^{*z}) that is the sum of (\bar{v}^z, \bar{w}^z) and a nondivergent velocity whose streamfunction is proportional to the meridional buoyancy flux:

$$(\mathbf{v}^{*z}, w^{*z}) = (\bar{v}^z, \bar{w}^z) + \hat{\mathbf{x}} \times \nabla_z \left[\frac{(\mathbf{v}'^z b'^z)^z}{\partial_z (\bar{b}^z)} \right]. \quad (\text{C1})$$

As a starting point for their TRM velocity, MM use Eq. (C1).

The fact that the residual circulation definition (C1) leads to a partial cancellation of the apparent diabatic term in the zonal mean buoyancy equation suggests a similar approach for the mean PV equation (2.15). We begin by defining a horizontal velocity,

$$\mathbf{u}_{\text{guess}}^{*y} = \bar{\mathbf{u}}^y + \hat{\mathbf{k}} \times \nabla \left[\frac{(\mathbf{u}'^y q')^y}{\partial_y (\bar{q}^y)} \right], \quad (\text{C2})$$

where the subscript “guess” indicates that this transformed velocity is defined heuristically by analogy with the zonal theory, and the gradient is understood to be taken in y coordinates. We here parallel MM’s approach for the three-dimensional mean buoyancy equation, starting with their Eq. (5). After extensive manipulations that use the eddy enstrophy equation (2.16), we find without approximation that

$$\begin{aligned} \frac{\partial \bar{q}^y}{\partial t} + \mathbf{u}_{\text{guess}}^{*y} \cdot \nabla \bar{q}^y \\ = \frac{\bar{D}^y}{Dt} \left\{ \frac{\partial}{\partial y} \left[\frac{1}{2} \frac{(\bar{q}')^{2^y}}{\partial_y (\bar{q}^y)} \right] \right\} + \mathbf{A} \cdot \nabla \bar{q}^y + S^{*y} + R_3 + R_4, \end{aligned} \quad (\text{C3})$$

where

$$\mathbf{A} = \hat{\mathbf{k}} \times \nabla \left\{ \frac{1}{2} \frac{\partial \bar{u}^y}{\partial y} \frac{(\bar{q}')^{2^y}}{[\partial_y (\bar{q}^y)]^2} \right\}, \quad (\text{C4})$$

$$S^{*y} = \bar{S}^y + \frac{\partial}{\partial y} \left\{ -\frac{(\bar{S}'^y q')^y}{\partial_y (\bar{q}^y)} + \frac{1}{2} \frac{\partial \bar{S}^y}{\partial y} \frac{(\bar{q}')^{2^y}}{[\partial_y (\bar{q}^y)]^2} \right\}, \quad (\text{C5})$$

and R_3 and R_4 involve higher-order products of the perturbation terms

$$R_3 = \frac{\partial}{\partial y} \left[\frac{\mathbf{u}'^y \cdot \nabla (\bar{q}')^{2^y}}{2 \partial_y (\bar{q}^y)} \right] \quad (\text{C6})$$

and

$$R_4 = -\frac{\partial}{\partial y} \left\{ \frac{(\bar{q}')^{2^y}}{2 [\partial_y (\bar{q}^y)]^2} \partial_y \left[\nabla \cdot (\mathbf{u}'^y q')^y \right] \right\}. \quad (\text{C7})$$

(In this appendix, x and t derivatives are taken at fixed y .)

The first two terms on the right-hand side of Eq. (C3) are quadratic in the eddy amplitude and could therefore contribute significantly to the mean flow forcing even in the absence of eddy transience and dissipation. The situation improves with some redefinitions, again fol-

lowing MM. First, we redefine the mean velocity to include the \mathbf{A} term; that is, we set

$$\begin{aligned} \mathbf{u}^{*y} &= \mathbf{u}_{\text{guess}}^{*y} - \mathbf{A} \\ &= \bar{\mathbf{u}}^y + \hat{\mathbf{k}} \times \nabla \left\{ \left[\frac{(\mathbf{u}'^y q')^y}{\partial_y (\bar{q}^y)} \right] - \frac{1}{2} \frac{\partial \bar{u}^y}{\partial y} \frac{(\bar{q}')^{2^y}}{[\partial_y (\bar{q}^y)]^2} \right\}. \end{aligned} \quad (\text{C8})$$

Next, we define a modified mean PV q^{*y} by

$$q^{*y} = \bar{q}^y - \frac{\partial}{\partial y} \left[\frac{1}{2} \frac{(\bar{q}')^{2^y}}{\partial_y (\bar{q}^y)} \right]. \quad (\text{C9})$$

Then (C3) may be rewritten, without approximation,

$$\begin{aligned} \frac{\partial q^{*y}}{\partial t} + \mathbf{u}^{*y} \cdot \nabla q^{*y} \\ = S^{*y} + R_3 + R_4 + (\mathbf{u}^{*y} - \bar{\mathbf{u}}^y) \cdot \nabla (q^{*y} - \bar{q}^y). \end{aligned} \quad (\text{C10})$$

Here MM derive analogous equations, although they only include terms of up to second order in disturbance amplitude.

Comparing (C9) with (2.12), (C8) with (2.13), and (C5) with (2.14), it is evident that to second order in eddy amplitude, $q^{*y} = q^{*q}$, $\mathbf{u}^{*y} = \hat{\mathbf{k}} \times \nabla \psi^{*q}$ and $S^{*q} = S^{*y}$. Thus, Eqs. (2.11) and (C10) are equivalent in a term-by-term sense to this degree of approximation.

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