# Estimation of Nonlinear Energy Transfer Spectra by the Cross-Spectral Method

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## ABSTRACT

Spectral formulas are derived to compute nonlinear energy transfer spectra by use of the cross-spectral technique. Nonlinear product terms are calculated directly from dependent variables without using the conventional interaction Fourier coefficients. The proposed method of computation is simpler than the conventional method and is applicable not only to wavenumber spectra but also to frequency or wavenumber-frequency spectra. Nonlinear aliasing errors associated with this approach can be either neglected or completely eliminated by Fourier interpolation. An example of the application of this method to atmospheric waves is given.

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## 1. Introduction

Saltzman (1957) proposed a wavenumber spectral analysis of the energetics of atmospheric disturbances in order to study the generation and transfer of energy among different wavenumbers. This technique has been applied not only to observed data [for a review see Saltzman (1970)], but also to general circulation models (Manabe *et al.* 1970; Tenenbaum 1976; Baker and Kung, 1977). Saltzman (1957) formulated nonlinear energy transfer spectra based on the Fourier transformed nonlinear equations of motion which involve product sums (convolution) of the Fourier transform of dependent variables. This transformation, in principle, is the same as that of a spherical harmonics transform of nonlinear equations as proposed by Silberman (1954), Kubota (1959) and Platzman (1960).

Alternatively, Manabe *et al.* (1970) computed nonlinear energy transfer wavenumber spectra more simply by taking a Fourier transform of the product of dependent variables which are multiplied directly at the grid points. In principle, their "grid" method is the same as that of computing a spherical harmonics transform of nonlinear equations as proposed by Orszag (1970) to make spectral models efficient. Burrows (1976) computed two-dimensional energy transfer spectra using spherical harmonics by the grid method. Murakami (1978) further separated one-dimensional nonlinear energy transfer spectra, computed by the grid method, into stationary and transient, as well as zonal and wave parts.

On the other hand, Hayashi, (1971, 1973, 1977a,b, 1979a,b) proposed a method of computing wavenumber-frequency cross spectra by use of frequency cross-spectral techniques without using the two-dimensional Fourier transform method by Kao (1968) [see reviews by Tsay (1974) and Pratt (1976)]. In the present paper, we shall propose a method of computing nonlinear energy transfer spectra using the cross-spectral technique by generalizing the grid method. The advantage of the present method is that it is simpler than Saltzman's and is applicable not only to wavenumber spectra but also to frequency spectra and wavenumber-frequency spectra.

In Section 2 cross spectra are defined. Formulas for computing nonlinear transfer spectra of kinetic energy are given in Section 3. An example of its application is given in Section 4. Notations are given in Appendix A. Formulas for available potential energy spectra are given in Appendix B. Nonlinear aliasing error is discussed in Appendix C.

## 2. Cross spectra

In this section cross spectra are defined since these spectra are used to compute the nonlinear spectra discussed in Section 3.

### *a. Wavenumber cross spectra*

It is assumed that space-time series data $u(\lambda,t)$ and $v(\lambda,t)$ are cyclic and discrete in longitude. These series are represented by a space-Fourier series with discrete wavenumbers ($n$) and Nyquist wavenumber ($N$) as

$$u(\lambda,t) = \mathrm{Re} \sum_{n=0}^{N} U_n(t)e^{in\lambda} \tag{2.1a}$$

$$= \mathrm{Re} \sum_{n=0}^{N} [C_n^u(t) \cos n\lambda + S_n^u(t) \sin n\lambda], \tag{2.1b}$$

where

$$U_n(t) = C_n^u(t) - iS_n^u(t). \qquad (2.2)$$

In particular, $C_0^u = u_0$ (zonal mean) and $S_0^u = S_N^u = 0$.

The *sample* wavenumber cospectra $P_n(u,v)$ and quadrature spectra $Q_n(u,v)$ (see Jenkins and Watts, 1968, p. 209) are defined as

$$P_n(u,v) = \tfrac{1}{2}\mathrm{Re}\,(U_n^* V_n) \qquad (2.3a)$$

$$= \tfrac{1}{2}(C_n^u C_n^v + S_n^u S_n^v), \qquad (2.3b)$$

$$Q_n(u,v) = \tfrac{1}{2}\mathrm{Im}\,(U_n^* V_n) \qquad (2.4a)$$

$$= \tfrac{1}{2}(S_n^u C_n^v - C_n^u S_n^v), \qquad (2.4b)$$

where the asterisk in this section denotes the complex conjugate and the cospectra (2.3) should be doubled for $n = 0$ and $n = N$.

The sample cospectrum is interpreted as the spectrum of the sample covariance averaged over $\lambda$ as

$$\overline{u(\lambda)v(\lambda)}^{\lambda} = \sum_{n=0}^{N} P_n(u,v). \qquad (2.5)$$

The sample quadrature spectrum is interpreted as the cospectrum between $u$ and $v$ with a 90° phase shift. The following relation is useful:

$$P_n\left(u, \frac{\partial v}{\partial \lambda}\right) = -nQ_n(u,v). \qquad (2.6)$$

More generally, the sample cross spectra $R_n(u,v)$ are defined as

$$R_n(u,v) = P_n(u,v) + iQ_n(u,v). \qquad (2.7)$$

The sample cross spectra are interpreted as the spectrum of the sample cross covariance as

$$\overline{u(\lambda)v(\lambda+\lambda_1)}^{\lambda} = \mathrm{Re}\sum_{n=0}^{N} R_n(u,v)e^{in\lambda_1}. \qquad (2.8)$$

The *true* wavenumber cross spectra are defined as the ensemble average of the sample wavenumber cross spectra. In practice, this ensemble average can be replaced by the time average of the sample wavenumber spectra for an ergodic time series.

The time-averaged wavenumber cross spectra can be decomposed into stationary (time mean) and transient (deviation from time mean) parts as

$$\bar{P}_n(u,v) = P_n^s(u,v) + P_n^t(u,v), \qquad (2.9)$$

$$\bar{Q}_n(u,v) = Q_n^s(u,v) + Q_n^t(u,v), \qquad (2.10)$$

where the overbar without $\lambda$ denotes the time average. Here these parts are computed as

$$P_n^s(u,v) = \tfrac{1}{2}(\bar{C}_n^u \bar{C}_n^v + \bar{S}_n^u \bar{S}_n^v), \qquad (2.11)$$

$$Q_n^s(u,v) = \tfrac{1}{2}(\bar{S}_n^u \bar{C}_n^v - \bar{C}_n^u \bar{S}_n^v), \qquad (2.12)$$

$$P_n^t(u,v) = \bar{P}_n(u,v) - P_n^s(u,v), \qquad (2.13)$$

$$Q_n^t(u,v) = \bar{Q}_n(u,v) - Q_n^s(u,v). \qquad (2.14)$$

The transient wavenumber cross spectra can be interpreted as wavenumber-frequency cross spectra which are integrated over all the progressive and retrogressive frequencies. If a particular wavenumber is associated with a particular frequency band due to the dispersion relation, the transient wavenumber cross spectra can be interpreted as wavenumber-frequency cross spectra.

The wavenumber coherence $\mathrm{Coh}^t$ and phase difference $\mathrm{Ph}^t$ between transient waves $u$ and $v$ are defined as

$$[\mathrm{Coh}_n^t(u,v)]^2 = \frac{[P_n^t(u,v)]^2 + [Q_n^t(u,v)]^2}{P_n^t(u,u)\cdot P_n^t(v,v)}, \qquad (2.15)$$

$$\mathrm{Ph}_n^t(u,v) = \tan^{-1}[Q_n^t(u,v)/P_n^t(u,v)]. \qquad (2.16)$$

These transient wavenumber spectra can also be used to estimate the horizontal and vertical coherence and phase difference of transient waves.

### *b. Wavenumber cross bispectra*

If $v(\lambda)$ is expressed as a product of two sets of series $b(\lambda)$ and $c(\lambda)$ as

$$v(\lambda) = b(\lambda)c(\lambda), \qquad (2.17)$$

the Fourier transform of $v(\lambda)$ is related to that of $b(\lambda)$ and $c(\lambda)$ by the convolution theorem (see Jenkins and Watts, 1968, p. 44) as

$$V_n = \tfrac{1}{2}\sum_{m=-N}^{N-n} B_m^* C_{n+m}, \qquad (2.18)$$

where $B_n$, $C_n$ are the complex Fourier transform of $b(\lambda)$ and $c(\lambda)$, respectively.

Inserting (2.18) into (2.3a) gives

$$P_n(u,v) = \mathrm{Re}\sum_{m=-N}^{N-n} \tfrac{1}{4} U_n^* B_m^* C_{n+m} \qquad (2.19a)$$

$$= \mathrm{Re}\sum_{m=-N}^{N-n} R_{n,m}(u,b,c), \qquad (2.19b)$$

where $R_{n,m}(a,b,c)$ is the wavenumber cross bispectrum of three sets of space series $a$, $b$ and $c$ as defined by

$$R_{n,m}(a,b,c) = \tfrac{1}{4} A_n^* B_m^* C_{n+m}. \qquad (2.20)$$

$R_{n,m}(a,b,c)$ is also interpreted as the spectrum of the cross-bicovariance as follows [for frequency bispectra see Hasselman *et al.* (1963), Hinich and Clay, (1968) and Roden and Bendiner (1973)]

$$\overline{a(\lambda)b(\lambda+\lambda_1)c(\lambda+\lambda_2)} = \mathrm{Re}\sum_{n=0}^{N}\sum_{m=-N}^{N-n} R_{n,m}(a,b,c)e^{i(n\lambda_1+m\lambda_2)}. \qquad (2.21)$$

The relation (2.19) is the principle of Saltzman's (1957) formulation of nonlinear energy transfer spectra. In the present paper, however, we make use of wavenumber cospectra (2.3) directly instead of (2.19). The advantage of the present approach is that it is simpler to program and the wavenumber cospectra can be formally replaced by either frequency or wavenumber-frequency cospectra as described below.

### *c. Frequency cross spectra*

The *true* frequency cross spectra are defined as the ensemble average of the sample frequency cross spectra. However, unlike the time-averaged sample wavenumber spectra, the true frequency spectra are not directly replaced by unsmoothed discrete sample frequency spectra. These time spectra should be estimated by use of more refined methods (see Bendat and Piersol, 1971, p. 330) such as the lag correlation method, direct Fourier transform method, segment method and filtering method. By use of these time-spectral techniques, wavenumber-frequency cross spectra can also be computed through spectra formulas derived by Hayashi (1971).

## 3. Kinetic energy spectra

### *a. Nonlinear energy transfer spectra*

The equations of motion and continuity in spherical coordinates are written in flux form (see Appendix A for notations) as

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial uu}{\partial x} - \frac{\partial vu}{\partial y} - \frac{\partial \omega u}{\partial p} + \frac{\tan\theta}{r} uv \right] + 2\Omega \sin\theta v - \frac{\partial \phi}{\partial x} + F_u, \quad (3.1)$$

$$\frac{\partial v}{\partial t} = \left[ -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} - \frac{\partial \omega v}{\partial p} - \frac{\tan\theta}{r} uu \right] - 2\Omega \sin\theta u - \frac{\partial \phi}{r\partial \theta} + F_v, \quad (3.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0, \quad (3.3)$$

where

$$\frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{r \cos\theta \partial\lambda}, \quad (3.4a)$$

$$\frac{\partial(\ )}{\partial y} = \frac{\partial \cos\theta(\ )}{r \cos\theta \partial\theta}. \quad (3.4b)$$

The terms in the brackets in (3.1) and (3.2) are nonlinear terms due to advection and sphericity.

Kinetic energy per unit mass $K_n$ for the wavenumber (or frequency) $n$ is defined by

$$K_n = \tfrac{1}{2}[P_n(u,u) + P_n(v,v)], \quad (3.5)$$

where $P_n$ is the wavenumber (or frequency) cospectrum.

The time change of wavenumber (or frequency) kinetic energy is given by

$$\frac{\partial K_n}{\partial t} = N_n(a,bc) - P_n(\alpha,\omega) - \left[ \frac{\partial P_n(\phi,u)}{\partial x} + \frac{\partial P_n(\phi,v)}{\partial y} + \frac{\partial P_n(\phi,\omega)}{\partial p} \right] + [P_n(u,F_u) + P_n(v,F_v)]. \quad (3.6)$$

This equation has been derived by taking a cospectrum between $u$ (or $v$) and both sides of Eq. (3.1) [or (3.2)] and making use of (3.3). In case of wavenumber spectra (not frequency spectra), the third term on the right-hand side of (3.6) which involves the $x$ derivative vanishes.

The nonlinear kinetic energy transfer spectrum $N_n(a,bc)$ in (3.6) is defined by

$$\begin{aligned} N_n(a,bc) = &- \left[ P_n\left(u, \frac{\partial uu}{\partial x}\right) + P_n\left(v, \frac{\partial uv}{\partial x}\right) \right] \\ &- \left[ P_n\left(u, \frac{\partial uv}{\partial y}\right) + P_n\left(v, \frac{\partial vv}{\partial y}\right) \right] \\ &+ \frac{\tan\theta}{r} [P_n(u,uv) - P_n(v,uu)] \\ &- \left[ P_n\left(u, \frac{\partial u\omega}{\partial p}\right) + P_n\left(v, \frac{\partial v\omega}{\partial p}\right) \right]. \end{aligned} \quad (3.7)$$

Here the symbol $a$ represents $u$ or $v$, while $bc$ denotes a nonlinear product term.

A transfer of kinetic energy from or into wavenumber $n$ occurs when the wavenumber of $a$, $b$ and $c$ are respectively $n$, $(n \pm m)$ and $m$ [see Eq. (2.19)] as related by

$$n = (n \pm m) \mp m. \quad (3.8)$$

It can be proven that the following relation holds:

$$\sum_{n=0}^{N} N_n(a,bc) = -\frac{\partial}{\partial x}\left(u \frac{u^2+v^2}{2}\right)_0 - \frac{\partial}{\partial y}\left(v \frac{u^2+v^2}{2}\right)_0 - \frac{\partial}{\partial p}\left(\omega \frac{u^2+v^2}{2}\right)_0, \quad (3.9)$$

where the suffix 0 denotes the zonal (or time) mean. Thus, the nonlinear kinetic energy transfer spectra tend to vanish, when they are integrated over the whole atmosphere as well as all the wavenumbers (or frequencies) $0 \sim N$.

The nonlinear products in (3.7) are computed at grid points, and the derivatives are replaced by finite difference. In case of wavenumber spectra the first two terms on the right-hand side of (3.7) can be computed without explicitly computing the $x$ derivative by use of (2.6) as

$$-\left[P_n\left(u,\frac{\partial uu}{\partial x}\right)+P_n\left(v,\frac{\partial uv}{\partial x}\right)\right]$$

$$=\frac{n}{r\cos\theta}\left[Q_n(u,uu)+Q_n(v,uv)\right], \qquad (3.10)$$

where $Q_n$ is the wavenumber quadrature spectrum defined in Section 2.

### *b. Partition of spectra into wave-wave and wave-mean interaction parts*

The nonlinear energy transfer spectra $N_n(a,bc)$ can be partitioned into two parts as

$$N_n(a,bc)=\langle K_m\cdot K_n\rangle+\langle K_0\cdot K_n\rangle. \qquad (3.11)$$

Here $\langle K_m\cdot K_n\rangle$ is the transfer of energy into wavenumber (frequency) $n$ by interaction among different wavenumbers (frequencies) excluding 0, while $\langle K_0\cdot K_n\rangle$ is the transfer of energy into wavenumber (frequency) $n$ by interaction between the mean flow and wavenumber (frequency) $n$.

By definition, $\langle K_m\cdot K_n\rangle$ is given as

$$\langle K_m\cdot K_n\rangle=N_n(a,b'c'), \qquad (3.12a)$$

where

$$N_n(a,b'c')=-\left[P_n\left(u,\frac{\partial u'u'}{\partial x}\right)+P_n\left(v,\frac{\partial u'v'}{\partial x}\right)\right]$$

$$-\left[P_n\left(u,\frac{\partial u'v'}{\partial y}\right)+P_n\left(v,\frac{\partial v'v'}{\partial y}\right)\right]$$

$$+\frac{\tan\theta}{r}\left[P_n(u,u'v')-P_n(v,u'u')\right]$$

$$-\left[P_n\left(u,\frac{\partial u'\omega'}{\partial p}\right)+P_n\left(v,\frac{\partial v'\omega'}{\partial p}\right)\right]. \qquad (3.12b)$$

Here, the prime denotes deviation from the zonal mean (or time mean) for wavenumber (or frequency) spectra.

It can be proven that the following relation holds as in (3.9):

$$\sum_{n=1}^{N}\langle K_m\cdot K_n\rangle$$

$$=-\frac{\partial}{\partial x}\left(u'\frac{u'^2+v'^2}{2}\right)_0-\frac{\partial}{\partial y}\left(v'\frac{u'^2+v'^2}{2}\right)_0$$

$$-\frac{\partial}{\partial p}\left(\omega'\frac{u'^2+v'^2}{2}\right)_0. \qquad (3.13)$$

Thus the wave-wave interaction part $\langle K_m\cdot K_n\rangle$ tends to vanish when integrated over all the wavenumbers (or frequencies) $1\sim N$ and over the whole atmosphere.

The mean-wave interaction part $\langle K_0\cdot K_n\rangle$ is given as a residual by

$$\langle K_0\cdot K_n\rangle=N_n(a,bc)-N_n(a,b'c'). \qquad (3.14a)$$

This residual is not a small difference of large numbers and no roundoff error occurs (see Table 2 in Section 4).

Alternatively, the above part $\langle K_0\cdot K_n\rangle$ is given explicitly by linearized equations as

$$\langle K_0\cdot K_n\rangle$$

$$=-\left[\frac{\partial u_0}{\partial x}P_n(u,u)+\frac{\partial v_0}{\partial x}P_n(u,v)\right]$$

$$-\left[\frac{\partial u_0}{r\partial\theta}P_n(u,v)+\frac{\partial v_0}{r\partial\theta}P_n(v,v)\right]$$

$$-\frac{\tan\theta}{r}\left[u_0P_n(u,v)-v_0P_n(u,u)\right]$$

$$-\left[\frac{\partial u_0}{\partial p}P_n(u,\omega)+\frac{\partial v_0}{\partial p}P_n(v,\omega)\right]$$

$$-\left[\frac{\partial u_0K_n}{\partial x}+\frac{\partial v_0K_n}{\partial y}+\frac{\partial\omega_0K_n}{\partial p}\right], \qquad (3.14b)$$

where the suffix 0 denotes zonal mean for wavenumber spectra and time mean for frequency spectra.

The formula (3.14a) is more convenient for computation than (3.14b), since the wave-mean interaction spectra can be computed in the same manner as the wave-wave interaction spectra. It should be noted that the last three terms of (3.14b) are interpreted as the convergence of the wave kinetic energy advected by the mean flow and are usually distinguished from the rest of the terms. In Saltzman's (1957) formulation these terms are implicitly contained in wave-wave interaction terms rather than in zonal-wave interaction terms. These three terms, however, tend to vanish when integrated over the entire atmosphere. In case of wavenumber spectra, terms in (3.14b) involving the $x$ derivative vanish.

It should be noted that for $n=0$, (3.11) is written as

$$N_0(a,bc)=\langle K_m\cdot K_0\rangle+\langle K_0\cdot K_0\rangle, \qquad (3.15)$$

where $\langle K_m\cdot K_0\rangle$ is given by putting $n=0$ in (3.12b) and $\langle K_0\cdot K_0\rangle$ is defined by

$$\langle K_0\cdot K_0\rangle=-\frac{\partial u_0K_0}{\partial x}-\frac{\partial v_0K_0}{\partial y}-\frac{\partial\omega_0K_0}{\partial p}. \qquad (3.16)$$

Here $\langle K_m\cdot K_0\rangle$ is interpreted as the transfer of kinetic energy into the mean flow by interaction between the mean flow and wave, while $\langle K_0\cdot K_0\rangle$ is the convergence of kinetic energy of the mean flow advected by the mean flow itself.

By virtue of (3.11) and (3.15), we have the relation

$$\langle K_m \cdot K_0 \rangle = -\sum_{n=1}^{N} \langle K_0 \cdot K_n \rangle + [\sum_{n=0}^{N} N_n(a,bc) - \sum_{n=1}^{N} \langle K_m \cdot K_n \rangle - \langle K_0 \cdot K_0 \rangle]. \quad (3.17)$$

Here, the last three terms are given explicitly by (3.9), (3.13) and (3.16) and tend to vanish when integrated over the entire atmosphere.

### *c. Partition of wavenumber spectra into stationary and transient parts*

The time-averaged wavenumber nonlinear energy transfer spectra can be further partitioned into stationary (time mean) and transient (deviation from time mean) parts as

$$\overline{\langle K_m \cdot K_n \rangle} = \langle K_m \cdot K_n^s \rangle + \langle K_m \cdot K_n^t \rangle, \quad (3.18)$$

$$\overline{\langle K_0 \cdot K_n \rangle} = \langle K_0 \cdot K_n^s \rangle + \langle K_0 \cdot K_n^t \rangle, \quad (3.19)$$

where the overbar denotes a time mean. These parts are interpreted as follows:

| | |
|---|---|
| $\langle K_m \cdot K_n^s \rangle$ | Transfer of kinetic energy into stationary wavenumber $n$ by interaction between different stationary waves or between different transient waves |
| $\langle K_m \cdot K_n^t \rangle$ | Transfer of kinetic energy into transient wavenumber $n$ by interaction between transient waves and stationary (or transient) waves |
| $\langle K_0 \cdot K_n^s \rangle$ | Transfer of kinetic energy into stationary wavenumber $n$ by interaction between stationary zonal mean flow and stationary wavenumber $n$ or between transient zonal mean flow and transient wavenumber $n$. |
| $\langle K_0 \cdot K_n^t \rangle$ | Transfer of kinetic energy into transient wavenumber $n$ by interaction between stationary zonal mean flow and transient wavenumber $n$ or between transient zonal mean flow and stationary (or transient) wavenumber $n$. |

These parts are computed as

$$\langle K_m \cdot K_n^s \rangle = N_n^s(a,b'c'), \quad (3.20)$$

$$\langle K_m \cdot K_n^t \rangle = N_n^t(a,b'c'), \quad (3.21)$$

$$\langle K_0 \cdot K_n^s \rangle = N_n^s(a,bc) - N_n^s(a,b'c'), \quad (3.22)$$

$$\langle K_0 \cdot K_n^t \rangle = N_n^t(a,bc) - N_n^t(a,b'c'). \quad (3.23)$$

Here, the prime denotes deviation from the zonal mean. $N_n^s$ and $N_n^t$ are defined by (3.7) and (3.12) with their cross spectra replaced by (2.11)–(2.14).

In addition to the above parts, it will also be useful to define the following spectra:

$$\langle K_m^s \cdot K_n^s \rangle = N_n^s(a,\bar{b}'\bar{c}'), \quad (3.24)$$

$$\langle K_m^t \cdot K_n^t \rangle = N_n^t(a,b'^*c'^*), \quad (3.25)$$

where the overbar denotes a time mean and the asterisk a deviation from the time mean.

These spectra are interpreted as follows:

| | |
|---|---|
| $\langle K_m^s \cdot K_n^s \rangle$ | Transfer of kinetic energy into stationary wavenumber by interaction *only* between different stationary wavenumbers. |
| $\langle K_m^t \cdot K_n^t \rangle$ | Transfer of kinetic energy into transient wavenumber $n$ by interaction *only* between different transient wavenumbers. |

These spectra correspond to steady and transient

TABLE 1. Simulated and observed wavenumber kinetic energy spectra ($10^5$J m$^{-2}$), wave-wave interaction part (W m$^{-2}$), and zonal-wave interaction part (W m$^{-2}$) as a function of wavenumber $n$. These spectra are integrated between the 100 and 1000 mb levels and averaged over the entire Northern Hemisphere during January.

| | $K_n$ | | $\langle K_m \cdot K_n \rangle$ | | $\langle K_0 \cdot K_n \rangle$ | |
|---|---|---|---|---|---|---|
| $n$ | Simulated | Observed | Simulated | Observed | Simulated | Observed |
| 1 | 2.0 | 1.1 | 0.09 | 0.09 | −0.19 | −0.01 |
| 2 | 0.8 | 1.1 | −0.12 | −0.03 | −0.04 | −0.06 |
| 3 | 0.7 | 1.3 | −0.05 | 0.16 | −0.10 | −0.03 |
| 4 | 0.7 | 0.5 | 0.03 | −0.07 | −0.05 | −0.01 |
| 5 | 0.6 | 0.6 | −0.02 | 0.05 | −0.13 | 0.00 |
| 6 | 0.4 | 0.7 | 0.02 | 0.07 | −0.08 | −0.07 |
| 7 | 0.3 | 0.4 | 0.00 | −0.14 | −0.06 | 0.03 |
| 8 | 0.2 | 0.3 | −0.02 | −0.05 | −0.02 | −0.03 |
| 9 | 0.2 | 0.2 | −0.01 | −0.04 | −0.02 | 0.01 |
| 10 | 0.1 | 0.2 | −0.02 | −0.01 | −0.01 | 0.00 |
| 11 | 0.1 | 0.1 | 0.00 | −0.01 | −0.01 | 0.01 |
| 12 | 0.1 | 0.1 | 0.00 | 0.00 | 0.00 | 0.01 |
| 13 | 0.05 | 0.1 | 0.02 | −0.02 | 0.00 | 0.00 |
| 14 | 0.04 | 0.04 | 0.01 | 0.01 | 0.00 | 0.01 |
| 15 | 0.03 | 0.04 | 0.01 | 0.02 | 0.00 | 0.00 |

TABLE 2a. As in Table 1 except for simulated stationary and transient wavenumber kinetic energy spectra. These spectra (January mean) are integrated between the 100 and 1000 mb levels and averaged over 30–60°N.

| | $K_n$ | | $\langle K_m \cdot K_n \rangle$ | | $\langle K_0 \cdot K_n \rangle$ | |
|---|---|---|---|---|---|---|
| $n$ | Stationary | Transient | Stationary | Transient | Stationary | Transient |
| 1 | 2.43 | 0.31 | 0.54 | −0.05 | −0.21 | −0.01 |
| 2 | 0.37 | 0.66 | −0.04 | −0.01 | −0.07 | 0.05 |
| 3 | 0.56 | 0.55 | −0.02 | 0.09 | −0.13 | −0.04 |
| 4 | 0.35 | 0.89 | −0.11 | −0.05 | −0.04 | 0.03 |
| 5 | 0.29 | 0.66 | 0.01 | 0.06 | −0.07 | −0.09 |
| 6 | 0.04 | 0.53 | 0.02 | −0.02 | −0.02 | −0.04 |
| 7 | 0.02 | 0.53 | −0.02 | −0.07 | 0.00 | −0.09 |
| 8 | 0.01 | 0.28 | −0.01 | −0.09 | −0.00 | −0.01 |
| 9 | 0.01 | 0.27 | 0.00 | −0.06 | 0.00 | −0.03 |
| 10 | 0.01 | 0.15 | −0.01 | −0.01 | −0.00 | −0.00 |
| 11 | 0.00 | 0.13 | −0.00 | −0.04 | −0.00 | −0.00 |
| 12 | 0.00 | 0.09 | −0.00 | −0.02 | 0.00 | −0.00 |
| 13 | 0.00 | 0.06 | −0.00 | 0.03 | −0.00 | 0.00 |
| 14 | 0.00 | 0.05 | 0.00 | −0.00 | 0.00 | −0.00 |
| 15 | 0.00 | 0.03 | −0.00 | 0.00 | 0.00 | −0.00 |

wave-wave interaction terms defined by Kanamitsu *et al.* (1972).

### *d. Computational procedure*

Computational procedure of nonlinear wavenumber energy transfer spectra is as follows:

1) Compute nonlinear product at grid points and compute the $y$, $p$ derivatives by finite difference. In case of wavenumber spectra, it is not necessary to compute the $x$ derivative explicitly. Compute $N_n(a,bc)$ by the formula (3.7) with (3.10), (2.3) and (2.4).

2) Subtract the zonal mean and compute nonlinear products. Compute $\langle K_m \cdot K_n \rangle$ by the formula (3.12b) with (3.10), (2.3) and (2.4). The computer code can be checked by use of the identity (3.13).

3) Compute $\langle K_0 \cdot K_n \rangle$ by the formula (3.14a) or (3.14b).

4) These spectra are further partitioned into stationary and transient parts by use of (3.20)–(3.25) with (2.11)–(2.14).

## 4. Example of application

As an example of the application of the present method, a wavenumber-spectral analysis is made of a spectral GFDL general circulation model (Manabe *et al.* 1979). This model has 30 zonal wavenumbers and the nonlinear term is computed at grid points following Orszag (1970). For a wavenumber analysis the values on the constant sigma surface are interpolated to the constant pressure surface and the nonlinear terms are recomputed at 96 grid points in longitude. This number of points is large enough to avoid nonlinear aliasing error (see Appendix C).

The wavenumber spectra are vertically mass-integrated and averaged over latitudes $\theta_1 \sim \theta_2$ as

$$\frac{\iiint K_n(dp/g)(rd\theta)(r\cos\theta d\lambda)}{\iint (rd\theta)(r\cos\theta)d\lambda} = \frac{\iint K_n dp(\cos\theta d\theta)}{g(\sin\theta_2 - \sin\theta_1)}. \quad (4.1)$$

This gives kinetic energy in a vertical column per unit area.

Table 1 shows the simulated and observed wavenumber spectral distribution of kinetic energy $K_n$, wave-wave interaction part $\langle K_m \cdot K_n \rangle$, and zonal-wave interaction part $\langle K_0 \cdot K_n \rangle$ integrated through the troposphere and averaged over the entire Northern Hemisphere. The simulated spectra are calculated by the present method using (3.5), (3.12b) and (3.14a), while the observed spectra were calculated (Tenenbaum, 1976) by the conventional method from NMC observational analysis data in January 1973. The simulated and observed spectra agree in the following respects: The kinetic energy decreases as the wavenumbers increase. Wavenumber 1 gains kinetic energy from higher wavenumbers as found observationally by Saltzman (1970). The low wavenumbers lose kinetic energy by zonal-wave interaction. However, there are some discrepancies between the simulated and observed interactions, particularly in $\langle K_m \cdot K_n \rangle$ for wavenumbers 2, 3 and 7 and in $\langle K_0 \cdot K_n \rangle$ for wavenumbers 1, 3 and 5.

Table 2 shows stationary and transient parts of simulated wavenumber spectra as calculated by (2.11)–(2.14) and (3.20)–(3.23). These spectra are further split into midlatitudes (30–60°N) and the tropics (0–30°N) as shown by Tables 2a and 2b, re-

TABLE 2b. As in Table 2a except that the spectra are averaged over 0–30°N.

| | $K_n$ | | $\langle K_m \cdot K_n \rangle$ | | $\langle K_0 \cdot K_n \rangle$ | |
|---|---|---|---|---|---|---|
| $n$ | Stationary | Transient | Stationary | Transient | Stationary | Transient |
| 1 | 1.62 | 0.36 | −0.19 | 0.02 | −0.29 | 0.06 |
| 2 | 0.31 | 0.31 | −0.21 | 0.02 | −0.10 | 0.04 |
| 3 | 0.24 | 0.31 | −0.14 | 0.00 | −0.12 | 0.03 |
| 4 | 0.17 | 0.37 | 0.07 | 0.09 | 0.02 | −0.11 |
| 5 | 0.15 | 0.35 | −0.04 | −0.04 | −0.04 | −0.11 |
| 6 | 0.08 | 0.30 | 0.02 | 0.02 | −0.01 | −0.12 |
| 7 | 0.03 | 0.22 | 0.00 | 0.05 | −0.02 | −0.05 |
| 8 | 0.02 | 0.15 | 0.00 | 0.03 | −0.00 | −0.02 |
| 9 | 0.01 | 0.12 | 0.00 | 0.01 | −0.01 | −0.02 |
| 10 | 0.01 | 0.09 | −0.00 | 0.03 | −0.00 | −0.01 |
| 11 | 0.01 | 0.07 | −0.01 | 0.02 | 0.00 | −0.00 |
| 12 | 0.01 | 0.06 | −0.00 | 0.02 | 0.00 | −0.00 |
| 13 | 0.00 | 0.04 | 0.00 | 0.02 | 0.00 | 0.00 |
| 14 | 0.00 | 0.04 | −0.00 | 0.02 | −0.00 | −0.00 |
| 15 | 0.00 | 0.03 | 0.00 | 0.01 | 0.00 | 0.00 |

spectively. In both the regions, stationary wavenumber 1 has the largest kinetic energy. Unlike midlatitude waves, however, the stationary wavenumbers 1–3 in the tropics lose their kinetic energy to higher wavenumber transient waves as found by Kanamitsu *et al.* (1972) observationally in the northern summer season.

## 5. Remarks

A method is proposed for computing nonlinear energy transfer spectra by use of cross-spectral techniques. Nonlinear product terms are calculated directly from dependent variables without using the conventional interaction Fourier coefficients. This method is simpler than the conventional method and is applicable not only to wavenumber spectra but also frequency or wavenumber-frequency spectra.

It should be noted that this grid method is, in principle, associated with a nonlinear aliasing error (see Appendix C), while the Fourier transform method (Saltzman, 1957) is free from this error. If data are generated by a numerical spectral model with the transform method (Orszag, 1970), the grid method is also free from this error, since the number of grid points of the model is large enough to eliminate the nonlinear aliasing error. In order to estimate the magnitude of nonlinear aliasing error, the computation in Section 4 was repeated with 60 grid points to resolve only 30 wavenumbers instead of 96 points used in the model. It was found that the error is less than 0.1%. This error can be neglected or completely eliminated by Fourier interpolation of grid data as described in Appendix C.

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## APPENDIX A

## List of Symbols

| | |
|---|---|
| $\Omega$ | angular velocity of earth |
| $r$ | radius of spherical earth 6371 km |
| $\lambda$ | longitude |
| $\theta$ | latitude |
| $p$ | pressure |
| $\dfrac{\partial(\ )}{\partial x}$ | $\dfrac{\partial(\ )}{r \cos\theta \partial\lambda}$ |
| $\dfrac{\partial(\ )}{\partial y}$ | $\dfrac{\partial \cos\theta(\ )}{r \cos\theta \partial\theta}$ |
| $u$ | eastward velocity |
| $v$ | northward velocity |
| $\phi$ | geopotential |
| $\omega$ | vertical pressure velocity |
| $T$ | temperature |
| $\alpha$ | specific volume |
| $J$ | heating per unit mass |
| $F$ | dissipation per unit mass |
| $R$ | gas constant ($=2.8704 \times 10^2$ m$^2$ S$^{-2}$ K$^{-1}$) |
| $\kappa$ | $1 - C_v/C_p$ ($= 2/7$) |
| $(\ )'$ | deviation from zonal mean |
| $(\ )^*$ | deviation from time mean |
| $\overline{(\ )}$ | time mean |
| $K_n$ | kinetic energy per unit mass for wavenumber (or frequency) $n$. |
| $A_n$ | available potential energy per unit mass for wavenumber (or frequency) $n$. |

## APPENDIX B

## Available Potential Energy Spectra

### 1. Wavenumber (frequency) energy equation

The thermodynamic equation is given by

$$\frac{\partial T}{\partial t} = -\frac{\partial uT}{\partial x} - \frac{\partial vT}{\partial y} - \left(\frac{\partial}{\partial p} - \frac{\kappa}{p}\right)\omega T + J/C_p. \quad \text{(B1)}$$

The available potential energy (Lorenz, 1955) associated with wavenumber (or frequency) $n$ ($n \neq 0$) is given approximately by

$$A_n = \tfrac{1}{2}\nu P_n(T), \quad \text{(B2)}$$

where

$$\nu = \frac{R}{p[\kappa \bar{T}^H/p - \partial \bar{T}^H/\partial p]}\,. \quad \text{(B3)}$$

Here, $\bar{T}^H$ is temperature averaged over globe and time at a single level defined by

$$\bar{T}^H = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \bar{T}_0 \cos\theta d\theta. \quad \text{(B4)}$$

For alternative formulations of available potential energy, see Boer (1975) and Pearce (1978).

The wavenumber (or frequency) available potential energy equation is given by

$$\frac{\partial A_n}{\partial t} = \langle A \cdot A_n \rangle + (\nu/p) C_n(T,J), \quad \text{(B5)}$$

$$N_n(a,bc) = -\nu\left\{P_n\left[T, \frac{\partial uT}{\partial x}\right] + P_n\left[T, \frac{\partial vT}{\partial y}\right] + P_n\left[T, \left(\frac{\partial}{\partial p} - \frac{\kappa}{p}\right)\omega T\right]\right\}. \quad \text{(B6)}$$

In case of wavenumber spectra the first term on the right-hand side of (B6) can be computed more conveniently by making use of (2.6) as

$$-\nu P_n\left(T, \frac{\partial uT}{\partial x}\right) = \frac{\nu n}{r\cos\theta} Q_n(T,uT). \quad \text{(B7)}$$

### 2. Partition of spectra into wave-wave and wave-mean interaction parts

The spectra $N_n(a,bc)$ can be partitioned into three parts as

$$N_n(a,bc) = \langle A_m \cdot A_n \rangle + \langle A_0 \cdot A_n \rangle + P_n(\alpha,\omega), \quad \text{(B8)}$$

where $P_n(\alpha,\omega)$ is a conversion from wave kinetic energy [see Eq. (3.6)]. These parts are defined as

$$\langle A_m \cdot A_n \rangle = N_n(a,b'c'), \quad \text{(B9)}$$

$$\langle A_0 \cdot A_n \rangle = N_n(a,bc) - N_n(a,b'c') - P_n(\alpha,\omega) \quad \text{(B10a)}$$

or equivalently

$$\langle A_0 \cdot A_n \rangle = -\nu \frac{\partial T_0}{\partial x} P_n(u,T) - \nu \frac{\partial T_0}{r\partial\theta} P_n(v,T) + \left\{\nu\left(\frac{\kappa T_0}{p} - \frac{\partial T_0}{\partial p}\right) - \frac{R}{p}\right\} P_n(\omega,T) - \frac{\partial u_0 A}{\partial x} - \frac{\partial v_0 A}{r\partial\theta} - \frac{\partial \omega_0 A}{\partial p} + \left(\frac{1}{\nu} \cdot \frac{\partial \nu}{\partial p} + \frac{2\kappa}{p}\right)\omega_0 A. \quad \text{(B10b)}$$

Here, the prime denotes deviation from the zonal mean for wavenumber spectra and from the time mean for frequency spectra. The suffix 0 in (B10b) denotes zonal mean for wavenumber spectra and time mean for frequency spectra. For wavenumber spectra, terms in (B10b) involving $x$ derivative vanish.

The third term in (B10b) vanishes if $\bar{T}^H$ coincides with $T_0$. The last term is usually neglected.

## APPENDIX C

## Nonlinear Aliasing Error

Due to nonlinear interaction among wavenumbers $0 \sim N$ ($N$ = Nyquist wavenumber, $2N$ = number of grid points), wavenumbers $0 \sim 2N$ are produced. However, due to aliasing error (see Bendat and Piersol, 1971, p. 228) wavenumbers $N + 1 \sim 2N$ are regarded as wavenumbers $N - 1 \sim 0$ on the $2N$ grid points. Phillips (1959) first pointed out this nonlinear aliasing error and proposed to eliminate it by Fourier filtering the top half of wavenumber components before computing the nonlinear products. Later, Orszag (1971) pointed out that it is sufficient to Fourier filter only the top third, since only wavenumber $0 \sim 4N/3$ components are nonlinearly produced and wavenumber $N + 1 \sim 4N/3$ components are aliased into only $N - 1 \sim 2N/3$ components.

Since we do not wish to filter high wavenumbers, we can alternatively increase the number of grid points $2N$ to at least $3N + 1$ by Fourier interpolation as

$$u_i = \sum_{n=0}^{N} [C_n \cos(n\lambda_i) + S_n \sin(n\lambda_i)], \quad \text{(C.1)}$$

where $i$ denotes the new grid points and $C_n$ and $S_n$ are the cosine and sine coefficients given by the original data.

It should also be remarked that there is no nonlinear aliasing error in space covariance $\overline{u(\lambda)v(\lambda)}^{\lambda}$, since the average over space ($\lambda$) eliminates aliased wavenumber components.

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