# 3. Scaling principles and filtered models

## 3.1 Incompressibility

Ignoring gravity for the moment, consider a fluid with uniform density $\rho_0$ in a state of rest. Disturb this state with a perturbation that is of small enough amplitude that one can justify ignoring all terms that are quadratic in this amplitude. Assume also that the flow is adiabatic, so that

$$\frac{dp}{dt} = \partial p/\partial\rho|_s \frac{d\rho}{dt}. \tag{3.1}$$

With the perturbation denoted by a prime, and assuming that the perturbations are infinitesimal, the linearized equation of motion and equation for the conservation of mass become

$$\rho_0 \frac{\partial \boldsymbol{V}'}{\partial t} = -\nabla p' \tag{3.2}$$

$$\frac{\partial\rho'}{\partial t} = -\rho_0 \nabla\cdot\boldsymbol{V}' = \frac{1}{\partial p/\partial\rho|_s}\frac{\partial p'}{\partial t}. \tag{3.3}$$

Eliminating variables in favor of $p'$ leads to the wave equation

$$p'_{tt} = c_s^2 \nabla^2 p', \tag{3.4}$$

where $c_s^2 = \partial p/\partial\rho|_s$ and $c_s$ is the *speed of sound*. For an ideal gas, $c_s^2 = \gamma RT$, so that $c_s = 350\ \mathrm{m/s}$ in the case of $T = 300\ \mathrm{K}$. Sound speeds are $\approx 10^3\ \mathrm{m/s}$ in the ocean.

Acoustic waves have little dynamical significance in meteorology or oceanography. One would like to modify, or *filter*, the equations in such a way as to eliminate them, hopefully without distorting the remaining non-acoustic, or *anelastic*, part of the flow. The simplest procedure is to assume that the fluid is incompressible. In an *incompressible* fluid, pressure variations produce negligible variations in the volume, or density, of fluid particles, and one sets $\nabla\cdot\boldsymbol{V} = 0$.

Incompressibility is a reasonable approximation for a fluid in a gravitational field if (1) velocities and phase speeds are much smaller than $c_s$ and (2) vertical scales of interest are much smaller than the scale height. To see this, assume that we are fortunate to have estimates of the length, time and velocity scales ($L$, $\tau$ and $U$) of the flow. Start by considering a steady, adiabatic flow, for which

$$\begin{aligned} c_s^2(\boldsymbol{V}\cdot\nabla\rho) &= \boldsymbol{V}\cdot\nabla p \\ &= \rho\boldsymbol{V}\cdot\nabla(|\boldsymbol{V}|^2/2+\Phi). \end{aligned} \tag{3.5}$$

The first equality follows from 3.1; the second from the equation of motion. If we continue to ignore the gravitational field, we can make the estimate

$$\delta\rho/\rho \approx U^2/c_s^2 = M^2, \tag{3.6}$$

where $\delta\rho$ is the magnitude of the density variation following the flow, and $M$ is the *Mach number*. If flow speeds are much smaller than the speed of sound, then the fractional change in the density following the flow due to pressure fluctuations will be small. Conservation of mass then yields

$$\nabla \cdot \boldsymbol{V} \approx M^2 U L^{-1} . \tag{3.7}$$

The individual terms, such as $\partial u / \partial x$, that make up $\nabla \cdot \boldsymbol{V}$ are typically of order $UL^{-1}$, so there must be near perfect cancellation between these terms if $M \ll 1$, and one is justified in setting $\nabla \cdot \boldsymbol{V} = 0$ in the equation of motion under these conditions.

Re-introducing gravity, we see that the possibility exists of a balance between the first and third terms of 3.5. Setting $\Phi = gz$, we have the estimate

$$\delta\rho / \rho \approx Lg / c_s^2 . \tag{3.8}$$

$L$ is now a measure of the vertical displacement of the fluid particle. For the atmosphere, $c_s^2 / g = \gamma RT / g$ is referred to as the adiabatic scale height (being the local density scale height of a constant-$\Theta$ atmosphere) and is of the order of 10 km. As long as vertical displacements are only of the order of 1 km or less, and if $M \ll 1$, incompressibility is a good approximation. For example, the planetary boundary layer, the region adjacent to the ground where turbulence is generated by surface stress and surface heating, is typically 1-2 km thick, and studies of the dynamics of this layer generally assume incompressibility. On the other hand, a convective tower in the tropics that extends from the planetary boundary layer to the tropopause cannot be accurately modeled with the incompressible equations. (Question: what is the scale height of the ocean?)

This scaling argument for steady flows is still relevant when the two terms in the material derivative $\partial / \partial t$ and $\boldsymbol{V} \cdot \nabla$ are of the same order, as is the case in turbulent flows. However, as the example of a linear acoustic wave makes explicit, one must also account for the possibility that $\partial / \partial t$ is the dominant term in the material derivative. In such a linear flow, $U \ll c = L\iota^{-1}$, where we can think of $c$ as the phase speed of a wave. The student is encouraged to provide an argument that 1) this phase speed must be much less than the speed of sound for the incompressible assumption to be accurate, and 2) in the presence of gravity, the waves must also have a vertical scale smaller than the adiabatic scale height.

The simplest incompressible set of equations results from the additional assumption that the density $\rho$ is a constant, $\rho_0$. In this familiar case of a homogeneous fluid, the equations of motion and the constraint $\nabla \cdot \boldsymbol{V} = 0$ form a complete set of equations for $\boldsymbol{V}$ and $p$. The pressure is determined by taking the divergence of the equation of motion to form a Poisson equation for $p$.

A more general incompressible model, and one of central importance for oceanographic modeling and theory, can be obtained by assuming that density variations are not identically zero, but are small: $\rho = \rho_0 + \rho_1 ;\quad \rho_1 \ll \rho_0 = \text{constant}$. It is then useful to define a time-independent pressure distribution $p_0$ in hydrostatic balance with the constant density $\rho_0$: $\rho_0 \nabla\Phi = \nabla p_0$. The total pressure is then $p_0 + p_1$, and by dropping terms of higher than first order in $p_1$ and $\rho_1$, the equation of motion is replace by

$$\frac{d\boldsymbol{V}}{dt} = -2\Omega \times \boldsymbol{V} - (\rho_1 / \rho_0)\nabla\Phi - \rho_0^{-1} \nabla p_1 . \tag{3.9}$$

Since we continue to assume incompressibility, there is no conversion of kinetic+potential into internal energy. To maintain a consistent energetics one must ignore the effects of divergence in the first law of thermodynamics. Therefore, 2.21 for the conservation of entropy reduces to

$$ds/dt = c_v dT/dt = 0 \tag{3.10}$$

if there is no heating or diffusion. If the density perturbation is simply proportional to a temperature perturbation, $\rho' = -aT'$, $T = T' + T_0$, then 3.10 reduces to

$$d\rho/dt = 0. \tag{3.11}$$

It is important to realize that this relation is not obtained by setting $\nabla \cdot \boldsymbol{V} = 0$ in the equation for conservation of mass. If there are heat sources, or thermal diffusion, then there will be sources and sinks of $\rho$. The divergence of the flow must then be non-zero also, strictly speaking, if mass is to be conserved, but one assumes that these diabatic effects are sufficiently weak that this divergence can continue to be ignored.

We define the buoyancy as $b = -g\rho_1/\rho_0$. The *Boussinesq approximation* can now be written (in the absence of heating and friction) as

$$\frac{d\boldsymbol{V}}{dt} = -2\Omega \times \boldsymbol{V} + b\hat{\boldsymbol{z}} - \rho_0^{-1}\nabla p \tag{3.12}$$

$$\nabla \cdot \boldsymbol{V} = 0 \tag{3.13}$$

$$\frac{db}{dt} = 0, \tag{3.14}$$

where we have dropped the subscript on the pressure to simplify the notation. Conservation of energy is easily proven as before. It now takes the form

$$\frac{\partial}{\partial t}[\rho_0(|\boldsymbol{V}|^2/2 + \Phi)] = -\nabla \cdot [\rho_0 \boldsymbol{V}(|\boldsymbol{V}|^2/2 + \Phi + p/\rho_0)], \tag{3.15}$$

which may be compared with 2.37. When heating is applied, it acts as a direct source of buoyancy, and of potential energy, rather than internal energy as in a compressible flow.

## 3.2 Gravitational stability

Imagine a fluid at rest in hydrostatic balance with the density profile $\rho_0(z)$. Picture an infinitesimal parcel of fluid displaced upwards, by a small amount, from $z_0$ to $z_0 + \delta z$, without altering the pressure field (we return to this last assumption in section 3.3). The parcel will accelerate upwards if it finds itself in a region in which its density is less than that of its environment, for then the upward pressure-gradient force, which balances the environmental density, will be greater than the gravitational force on the parcel. The result will then be a growing, unstable parcel displacement.

Consider first the case of an incompressible flow, in which the density of a parcel is conserved. Instability will then occur if $\partial\rho_0/\partial z > 0$, if heavy fluid overlies light fluid. In the stable case, $\partial\rho_0/\partial z < 0$, the magnitude of the upward force will be $-g(\rho - \rho_0)$, or $g(\partial\rho_0/\partial z)\delta z$, where $\delta z$ is the vertical displacement. Therefore,

$$\frac{\partial^2}{\partial t^2}\delta z = -N^2 \delta z, \tag{3.16}$$

where $N^2 = -g\rho_0^{-1}\partial\rho_0/\partial z$. $N$ is the frequency at which the parcel will oscillate in the stable case, and is referred to as the Brunt-Vaisala frequency or *buoyancy frequency.*

In a compressible fluid, gravitational stability is still determined by the relative magnitude of the parcel and environmental densities, but it is the parcel's entropy, not its density, that is conserved as it is displaced. Consequently, for an ideal gas, $\Theta - \Theta_0 = -(\partial\Theta_0/\partial z)\delta z$. But if the pressure is unperturbed, then

$$(\Theta - \Theta_0)/\Theta_0 = (T - T_0)/T_0 = -(\rho - \rho_0)/\rho. \tag{3.17}$$

Therefore, 3.16 is unchanged except that $N^2$ is now defined to be $N^2 = g\Theta_0^{-1}\partial\Theta_0/\partial z$. It is the vertical gradient of the entropy, or the potential temperature, that determines the gravitational stability of the atmosphere. For a neutrally stable atmosphere, with $\partial\Theta_0/\partial z = 0$, in hydrostatic balance, one has $\partial T_0/\partial z = -g/c_p \approx -9.8$ K/km. If temperatures decrease more rapidly than this *dry adiabatic lapse rate*, so that $\partial\Theta_0/\partial z < 0$, then the atmosphere is gravitationally unstable.

In the ocean, we have a rather complicated equation of state, which we can write as $\rho = \rho(T, S, p)$ -- that is, density is a function of temperature, pressure, and salinity. (Upper-case $S$ is salinity, not entropy.) In high latitudes, near the freezing temperature, variations in $S$ tend to dominate -- in low latitudes, variations in $T$ are more important. The influence of pressure on density is larger than all of these effects if we vary the pressure from its surface value to that at the bottom of the ocean, but this compression has little dynamical significance. The fact that the way in which the density depends on $T$ and $S$ is itself dependent on pressure is of more importance. For example, the salty water flowing out of the Mediterranean is denser than the cold and fresh bottom water formed near the coast of Antarctica if we compare these waters at surface pressures, but the Antarctic waters are more dense at deep pressures. We can also find expressions for the entropy, or potential temperature $\Theta$ of seawater: $\Theta = \Theta(T, S, p)$. Even better, we can use this to express density as a function of $\Theta$ rather than $T$: $\rho = \rho(\Theta, S, p)$.

We define the *potential density*, $\mu$, as the density that a parcel of water would have if its pressure were changed to some reference pressure $p^*$ holding the entropy and density fixed -- i.e., $\mu = \rho(\Theta, S, p^*)$. There is a different potential density for each reference pressure. The potential density is conserved following the parcel if the entropy and the salinity are conserved. By an argument similar to that above, we can now show that the buoyancy frequency in the ocean is $N^2 = -g\mu_0^{-1}\partial\mu_0/\partial z$, where $\mu_0$ is the potential density of the unperturbed state, referenced to the pressure at which we are computing $N^2$. The complication in oceanography is that one cannot, in general, define one function of $z$ whose derivative provides the buoyancy frequency -- one needs to use the potential density referenced to the local pressure. See Problem 3.1.

The manner in which the large-scale gravitational stability is maintained is of central importance in both meteorology and oceanography, since its magnitude greatly affects the structure of the circulation.

For another perspective on gravitational instability, consider small disturbances to a Boussinesq fluid at rest with a uniform buoyancy gradient $b_z = -g\rho_0^{-1}\partial\rho_1/\partial z = N^2$, a constant. Ignore rotation and the spatial variation of $\hat{\boldsymbol{z}}$, and assume that the flow is a function of $x$ and $z$ only. The resulting *linear* equations are

$$\begin{aligned} u'_t &= -\rho_0^{-1}p'_x \\ w'_t &= +b' - \rho_0^{-1}p'_z \\ u'_x + w'_z &= 0 \\ b'_t &= -w'N^2 \,. \end{aligned} \tag{3.18}$$

Eliminating other variables in favor of the vertical velocity leads to the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2}w' + N^2\frac{\partial^2}{\partial x^2}w' = 0\,. \tag{3.19}$$

Since $N^2$ is independent of $z$, we can look for solutions of the form $\psi' = \mathrm{Re}\{\Psi\exp[i(kx + mz - \omega t)]\}$, leading to the dispersion relation for internal gravity waves:

$$\omega^2 = k^2N^2/(k^2 + m^2)\,. \tag{3.20}$$

The frequency $\omega$ is always less than the buoyancy frequency $N$, approaching $N$ for small horizontal scales ($k \gg m$). Note that the pressure perturbations have a negligible effect on the dynamics in this limiting case. This is consistent with the result that $\omega = N$ in the familiar parcel argument, which assumes explicitly that the pressure is unperturbed.

If $\partial\rho_1/\partial z > 0$, then

$$\omega = \pm ikn/(k^2 + m^2)^{1/2}, \tag{3.21}$$

where $n^2 = g\rho_0^{-1}(\partial\rho_1/\partial z)$. Since the flow has the dependence $\exp(-i\omega t)$, the positive sign in 3.21 corresponds to exponential growth, $\exp(\sigma t)$, with e-folding time

$$\sigma^{-1} = n^{-1}(1 + \xi^2)^{1/2}, \quad \xi = m/k\,. \tag{3.22}$$

The growth rate depends only on the ratio of the vertical and horizontal scales; in this sense there is no scale selection in gravitational instability (when viscosity and thermal conductivity are negligible). Maximum growth $\sigma = n$ occurs when horizontal scales are much smaller than vertical scales.

## 3.3 The anelastic approximation

The Boussinesq equations are not adequate for the study of deep motions in the atmosphere, with vertical scales comparable to the scale height. For such flows it is no longer valid to assume that density variations due to pressure gradient forces are negligible following the flow. A useful generalization of the Boussinesq system was provided by Ogura and Phillips. One still expresses all thermodynamic variables as small perturbations about a reference atmosphere, but now the density in this reference state is a function of height, $\rho_0(z)$. Potential temperature, and not density, is conserved following the flow, in the absence of heating. Ogura and Phillips consider the special case in which the reference atmosphere has a constant potential temperature $\Theta_0$. Assuming that $\Theta = \Theta_0 + \Theta_1$ and $\rho = \rho_0(z) + \rho_1$, etc., we can approximate the pressure gradient force as

$$\rho^{-1}\nabla p = c_p \Theta \nabla \Pi \approx c_p \Theta_0 \nabla \Pi_1 + c_p \Theta_1 \nabla \Pi_0 = b\hat{\boldsymbol{z}} + c_p \Theta_0 \nabla \Pi_1, \tag{3.23}$$

where the buoyancy is now equal to $b = g\Theta_1/\Theta_0$. The thermodynamic equation is simply $d\Theta_1/dt = 0$ for adiabatic flow, or $d\Theta_1/dt = Q/(c_p\Pi_0)$ in the presence of the heating $Q$ per unit mass.

The final equation needed to complete this set can be obtained by assuming that pressure variations following the fluid can be estimated from the pressure of the reference atmosphere: $dp/dt = \gamma p \rho^{-1} d\rho/dt = -\gamma p \nabla \cdot \boldsymbol{V} \approx -\gamma p_0 \nabla \cdot \boldsymbol{V}$. From the ideal gas law, one has

$$p_0^{-1}\partial p_0/\partial z = \gamma(\rho_0\Theta_0)^{-1}\partial(\rho_0\Theta_0)/\partial z, \tag{3.24}$$

so that $\nabla \cdot \boldsymbol{V} = -w(\rho_0\Theta_0)^{-1}\partial(\rho_0\Theta_0)/\partial z$, or $\nabla \cdot (\rho_0\Theta_0\boldsymbol{V}) = 0$. For a constant $\Theta_0$ reference atmosphere, and in the absence of heating and dissipation, the complete set of *anelastic* equations becomes

$$d\boldsymbol{V}/dt = -2\Omega \times \boldsymbol{V} + g(\Theta_1/\Theta_0)\hat{\boldsymbol{z}} - c_p\Theta_0\nabla\Pi_1 \tag{3.25}$$

$$d\Theta_1/dt = 0 \tag{3.26}$$

$$\nabla \cdot (\rho_0 \boldsymbol{V}) = 0, \tag{3.27}$$

where $\Pi_1$ can be thought of as determined implicitly, just as in the homogeneous-incompressible and Boussinesq systems.

The equations reduce to the Boussinesq system when attention is confined to a region in which the variation of the basic state density is negligible. Whenever one uses the Boussinesq equations to describe the atmosphere, one should think of them as this limiting case of the anelastic system, with the buoyancy equal to $g\Theta/\Theta_0$. (Can you derive energy conservation for this system?)

## 3.4. Linear waves in an isothermal atmosphere

It is informative to consider the dispersion relation for linear waves in an isothermal ($T = \text{const}$) atmosphere at rest, ignoring spherical geometry and rotation but without any further approximation. The unperturbed state is denoted by the subscript *r*. Since it is at rest, the unperturbed atmosphere is in hydrostatic balance:

$$\partial p_r / \partial z = -\rho_r g \,. \tag{3.28}$$

For simplicity assume that the motion is confined to the *x*-*z* plane. The linearized equations are

$$\begin{aligned} \rho_r \partial u' / \partial t &= -\partial p' / \partial x \\ \rho_r \partial w' / \partial t &= -\partial p' / \partial z - g\rho' \\ \partial \rho' / \partial t + w' \partial \rho_r / \partial z &= -\rho_r \partial w' / \partial z - \rho_r \partial u' / \partial x \\ \partial \Theta' / \partial t + w' \partial \Theta_r / \partial z &= 0 \\ p' / p_r &= \gamma \rho' / \rho_r + \gamma \Theta' / \Theta_r \,. \end{aligned} \tag{3.29}$$

The final equation can be obtained from the equation of state in the form $p = \rho R \Theta \Pi$, noting that $\gamma = 1/(1-\kappa)$.

This set can be reduced to two equations in the unknowns $u'$ and $w'$. In the special case of an isothermal basic state, we have $\rho_r = e^{-z/H}$, with $H = RT_r/g$, and $\Theta_r = T_r e^{\kappa z/H}$, while $c_s^2 = \gamma R T_r = \gamma g H$ is a constant. After some manipulation, one obtains for this special case

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2}\right) u' = c_s^2 \left(\frac{\partial}{\partial z} - \frac{1}{\gamma H}\right) \frac{\partial}{\partial x} w' \tag{3.30}$$

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \left(\frac{\partial^2}{\partial z^2} - \frac{1}{H}\frac{\partial}{\partial z}\right)\right) w' = c_s^2 \left(\frac{\partial}{\partial z} - \frac{\kappa}{H}\right) \frac{\partial}{\partial x} u' \,, \tag{3.31}$$

or, after a few final cancellations,

$$\frac{\partial^4}{\partial t^4} w' - c_s^2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{H}\frac{\partial}{\partial z}\right) w' - c_s^2 \kappa g H^{-1} \frac{\partial^2}{\partial x^2} w' = 0 \,. \tag{3.32}$$

The term with the single *z*-derivative can be removed by setting

$$w' = W e^{z/(2H)} \,. \tag{3.33}$$

Thus,

$$\frac{\partial^4}{\partial t^4} W - c_s^2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{4H^2}\right) W - c_s^2 \kappa g H^{-1} \frac{\partial^2}{\partial x^2} W = 0 \,. \tag{3.34}$$

This imposing equation has constant coefficients, and one may look for wavelike solutions of the form

$$W = \mathrm{Re}\{\tilde{W} e^{i(kx + mz - \omega t)}\} \,, \tag{3.35}$$

where (*k*,*m*) is the wavenumber in the (*x*,*z*) plane, $\omega$ is the frequency, and $\tilde{W}$ is the complex amplitude. The following dispersion relation for *acoustic-gravity* waves in an isothermal atmosphere results:

$$\omega^4 - c_s^2\omega^2\left(k^2 + m^2 + \frac{1}{4H^2}\right) + c_s^2 N^2 k^2 = 0. \tag{3.36}$$

The solution for $\omega$ is

$$\omega^2 = c_s^2 K^2 \left\{1 \pm [1 - 4N^2k^2/(c_s^2K^4)]^{1/2}\right\}/2, \tag{3.37}$$

where $K^2 = k^2 + m^2 + 1/(4H^2)$. The argument of the square root is minimized when $m^2 = 0$ and $k^2 = 1/(4H^2)$, whereupon it takes the value $1 - 4N^2H^2/c_s^2$. But $N^2H^2/c_s^2 = \kappa/\gamma \approx 0.2$, so the argument is always positive. It follows that all modes are stable ($\omega$ is real).

The positive sign in 3.37 corresponds to the high-frequency acoustic branch and the negative sign to the low-frequency gravity-wave branch. Acoustic and gravity waves are well separated if

$$4N^2k^2/(c_s^2K^4) \ll 1, \tag{3.38}$$

in which case

$$\omega_+^2 = c_s^2K^2 \qquad \text{and} \qquad \omega_-^2 = N^2k^2/K^2. \tag{3.39}$$

The condition 3.38 is satisfied for our isothermal atmosphere if

$$4\kappa\gamma^{-1}(kH)^2 \approx 0.8(kH)^2 \ll [(kH)^2 + (mH)^2 + 1/4]^2,$$

so it is only in the particular case that both $k$ and $m$ are of the order of $H^{-1}$ that this frequency separation is not sharp.

Note also that there are solutions of the set of equations 3.30-3.31 with $w' = 0$. They are of the form $u' = \mathrm{Re}\{Ue^{i(kx + mz - \omega t)}\}$, with $\omega = c_s k$ and $U(z) = e^{\kappa z/H}$. These horizontally propagating sound waves are referred to as *Lamb waves*. Even though their amplitude increases with height, they are external, or surface-trapped waves, since their kinetic energy density $\rho_0|U|^2$ decreases exponentially away from the surface. The kinetic energy density of the internal waves 3.35 is independent of height.

We have already seen that the Boussinesq equations in this same geometry produce the dispersion relation

$$\omega^2 = N^2k^2/(k^2 + m^2). \tag{3.40}$$

This corresponds to the gravity-wave branch of the full dispersion relation for the isothermal atmosphere, when wavelengths are sufficiently small that the term $(4H^2)^{-1}$ is negligible in the denominator.

## 3.5. Hydrostatic balance

The atmosphere and the oceans are both thin shells of fluid on a nearly spherical earth. Therefore, we are often interested in flows for which the aspect ratio $\alpha$ is small, where $\alpha$ is the ratio of the characteristic vertical scale $H$ over which the flow varies to the characteristic horizontal scale $L$. For an extratropical cyclone, with a characteristic horizontal scale of $10^6$ m, $\alpha \approx 10^{-2}$. When the aspect ratio is small, one can generally assume that the flow is in hydrostatic balance *even when it is not at rest.*

Consider first a non-rotating Boussinesq fluid. We assume that the horizontal and vertical length and velocity ($U$,$W$) scales are related by $U/L \approx W/H$, and that we can set $d/dt \approx U/L$ or $W/H$. From the horizontal equation of motion one can then estimate the magnitude of the pressure variations as $P \approx \rho_0 U^2$. Therefore,

$$\frac{|dw/dt|}{|\rho^{-1}\partial p'/\partial z|} \approx (W/U)^2 \approx \alpha^2 . \tag{3.41}$$

The same estimate is obtained if one assumes instead that the local time derivative dominates the material derivative in each equation. Therefore, when the aspect ratio is small, we expect to be able to ignore the material derivative in the vertical equation of motion, leaving hydrostatic balance. The resulting system of equations is (indicating the horizontal component of a vector by the subscript $h$)

$$\frac{d\boldsymbol{V}_h}{dt} = -\rho_0^{-1}\nabla_h p \tag{3.42}$$

$$\frac{d\rho}{dt} = 0 \tag{3.43}$$

$$\frac{\partial p}{\partial z} = -g\rho \tag{3.44}$$

$$\nabla_h \cdot \boldsymbol{V}_h + \frac{\partial w}{\partial z} = 0 , \tag{3.45}$$

where $d/dt = \partial/\partial t + \boldsymbol{V}_h \cdot \nabla + w\partial/\partial z$.

While this model may not appear much simpler than the full equations, in fact, the ability to think of the pressure at a point as being determined by the weight of the fluid above it is a simplification whose importance is hard to overestimate, as it colors all of our thinking about large-scale flows in the atmosphere and ocean.

Multiplying the horizontal equation of motion by $\rho_0 \boldsymbol{V}_h$ and using 3.44 and 3.45, we have

$$\rho_0 \frac{d(|\boldsymbol{V}_h|^2/2)}{dt} = -\boldsymbol{V}_h \cdot \nabla_h p = -\nabla_h \cdot (\boldsymbol{V}_h p) - p\frac{\partial w}{\partial z} = -\nabla \cdot (\boldsymbol{V}p) - \rho g w . \tag{3.46}$$

From this result we find that these hydrostatic equations retain an *exact* energy conservation law

$$\frac{\partial}{\partial t}(\rho_0 |\boldsymbol{V}_h|^2/2 + \Phi) = -\nabla \cdot [\boldsymbol{V}(\rho_0 |\boldsymbol{V}_h|^2/2 + \Phi + p)], \qquad \Phi = \rho g z , \tag{3.47}$$

that is identical to the full Boussinesq version, except that the kinetic energy of vertical motion must be omitted in the energy density.

One can easily repeat the analysis for a small disturbance to a Boussinesq fluid at rest with a uniform Brunt-Vaisala frequency, but using these hydrostatic equations. The only difference is the deletion of the term $\partial w' / \partial t$ in the vertical equation of motion, leading to the dispersion relation for hydrostatic gravity waves:

$$\omega^2 = k^2 N^2 / m^2 . \tag{3.48}$$

Consistent with the "derivation" of the hydrostatic model, this is the appropriate approximation to the non-hydrostatic dispersion relation when the aspect ratio is small ($k \ll m$).

If we consider the gravitationally unstable case, $N^2 < 0$, the hydrostatic dispersion relation produces the unphysical result that the maximum growth rate is unbounded. Clearly, it makes little sense to allow gravitational instability to occur in a hydrostatic system, as the most unstable disturbances in an inviscid flow have large aspect ratio ($k \gg m$). Yet, except in the simplest linear problems the development of gravitational instability from hydrostatic antecedents is common. An important example is that of a hydrostatic gravity wave that grows in amplitude as it propagates upwards (see Eq. ?), eventually producing overturning isentropes. For a problem such as this, and many others, the hydrostatic equations are not well posed, but require the addition of terms that mimic the effects of the non-hydrostatic, turbulent mixing induced by gravitational instability.

Compressibility adds no new complications (except that one should allow for the possibility that the characteristic vertical scale is larger than the scale height, in which case it should be replaced by the latter in the scale analysis.) The resulting model is

$$\frac{d\boldsymbol{V}_h}{dt} = -\rho^{-1}\nabla_h p \tag{3.49}$$

$$\frac{d\theta}{dt} = 0 \tag{3.50}$$

$$\frac{\partial p}{\partial z} = -g\rho \tag{3.51}$$

$$\frac{d\rho}{dt} = -\rho\left(\nabla_h \cdot \boldsymbol{V}_h + \frac{\partial w}{\partial z}\right). \tag{3.52}$$

After one has made the hydrostatic approximation, it is often convenient to change the vertical coordinate. A very common choice in meteorology is *pressure coordinates*. The hydrostatic relation ensures that pressure decreases monotonically with increasing height, so the transformation from $z$ to $p$ is well-defined. (It should be emphasized that by making the hydrostatic approximation, we have already distinguished between the horizontal and vertical directions, and $(u,v)$ continues to refer to the horizontal flow, perpendicular to the $z$-axis.) The "vertical velocity" in pressure coordinates is $dp/dt \equiv \omega$. From the definition of the material derivative, we have

$$d/dt = \partial/\partial t + \boldsymbol{V}_h \cdot \nabla_h + \omega\partial/\partial p , \tag{3.53}$$

where the horizontal and time derivatives are taken at constant pressure.

To convert from differentiation at constant $z$ to differentiation at constant $p$, the chain rule (or draw a figure) implies

$$\left.\frac{\partial}{\partial\xi}\right|_p = \left.\frac{\partial}{\partial\xi}\right|_z + \left.\frac{\partial p}{\partial z}\frac{\partial z}{\partial\xi}\right|_p \frac{\partial}{\partial p} = \left.\frac{\partial}{\partial\xi}\right|_z - \rho g \left.\frac{\partial z}{\partial\xi}\right|_p \frac{\partial}{\partial p}, \tag{3.54}$$

where $\xi$ is either *x*, *y* or *t*. Therefore, the pressure gradient in the horizontal equation of motion becomes

$$-\rho^{-1}\nabla_h p\big|_z = -g\nabla_h z = -\nabla_h\Phi, \tag{3.55}$$

where $\Phi$ is the geopotential and the derivatives are at constant pressure if not otherwise indicated. Away from the surface, routine weather observations, which form the basis of much of our understanding of large-scale atmospheric circulations, are invariably presented in pressure coordinates. A common “weather map” consists of a plot of the height of the 500 mb (50 hPa) pressure surface, rather than pressure at a certain height.

The pressure-coordinate form of mass conservation is most easily obtained from the expression for the mass element,

$$\delta m = \rho\delta x\delta y\delta z = -g^{-1}\delta x\delta y\delta p, \tag{3.56}$$

so that

$$(\delta m)^{-1}\frac{d(\delta m)}{dt} = 0 = (\delta x)^{-1}\frac{d(\delta x)}{dt} + (\delta y)^{-1}\frac{d(\delta y)}{dt} + (\delta p)^{-1}\frac{d(\delta p)}{dt}. \tag{3.57}$$

Since $\delta u = d(\delta x)/dt$, $\delta v = d(\delta y)/dt$ and $\delta\omega = d(\delta p)/dt$, the result is

$$\nabla_h \cdot \boldsymbol{V}_h + \partial\omega/\partial p = 0. \tag{3.58}$$

Mass conservation for hydrostatic flow in pressure coordinates takes a form analogous to that in an incompressible fluid. It is this simplification, along with the simpler expression for the pressure gradient 3.55, that provides the motivation for this change in coordinate. The only complication arises from the lower boundary condition. Since

$$w = dz/dt = \partial z/\partial t|_p + \boldsymbol{V}_h \cdot \nabla_h z\big|_p + \omega\partial z/\partial p, \tag{3.59}$$

the condition of no normal flow on a flat surface ($w = 0$ at $z = 0$) translates in pressure coordinates into

$$\omega = -\rho(\Phi_t + \boldsymbol{V}_h \cdot \nabla_h\Phi) \quad \text{at} \quad p = p(x, y, 0, t). \tag{3.60}$$

It is sometimes convenient to stretch the pressure coordinate to mimic a height coordinate, by setting $Z = -H\ln(p/p_*)$. $H$ is a constant included to give $Z$ the units of length. We set it equal to $RT_0/g$, where $T_0$ is a reference temperature, and also define $\Theta_0(p) = T_0\Pi$. The equations of motion in this log(pressure) coordinate are

$$\frac{d\boldsymbol{V}_h}{dt} = -\nabla_h \Phi$$

$$\frac{d\Theta}{dt} = 0 \tag{3.61}$$

$$\frac{\partial \Phi}{\partial Z} = gT/T_0 = g\Theta/\Theta_0,$$

where $W \equiv dZ/dt$ and $d/dt = \partial/\partial t + \boldsymbol{V}_h \cdot \nabla_h + W\partial/\partial Z$, with the lower boundary condition

$$W = g^{-1}(\Phi_t + \boldsymbol{V}_h \cdot \nabla_h \Phi). \tag{3.62}$$

This coordinate is very common in discussions of stratospheric dynamics. Consider a basic state resting atmosphere with the stratification $\Theta = \Theta_r(Z)$. Infinitesimal 2-dimensional ($v = 0$) perturbations about this state satisfy

$$u'_t = -\Phi_x$$

$$\Theta'_t = -W'(d\Theta_r/dZ) \tag{3.63}$$

$$u'_x + p^{-1}(pW')_Z = 0,$$

plus the hydrostatic equation. These can be combined into

$$(p^{-1}(pW')_Z)_{Ztt} = -\underline{N}^2 W'_{xx}. \tag{3.64}$$

Here $\underline{N}^2 \equiv g\Theta_0^{-1}(d\Theta_r/dZ)$. (Note that this is not precisely the Brunt-Vaisala frequency unless the atmosphere is isothermal). In the case that the reference state being perturbed is isothermal, $\underline{N}^2 = N^2 = \kappa g/H$ is a constant. Setting

$$W = (p/p_*)^{-1/2}\mathrm{Re}\{We^{i(kx + mZ - \omega t)}\}, \tag{3.65}$$

we have the dispersion relation

$$\omega^2 = \underline{N}^2 k^2/(m^2 + (2H)^{-2}). \tag{3.66}$$

This is in agreement with the gravity-wave branch of the full dispersion relation for an isothermal atmosphere, given that the aspect ratio is small. By omitting the acceleration of the vertical velocity, the hydrostatic approximation has eliminated all acoustic oscillations -- except for the purely horizontally propagating Lamb wave -- without requiring any assumption about incompressibility.

It is informative to show that the Lamb wave can be retrieved from these log(pressure) equations, by applying the linearized boundary condition, $W' = g^{-1}\Phi'_t$ at $Z = 0$ and looking for solutions that decay for large $Z$.

The choice of $\Pi$ as vertical coordinate also have some admirers, since it results in a hydrostatic equation ($\partial\Phi/\partial\Pi = -c_p\theta$) that is closer in form to that in a Boussinesq model, with the “vertical” derivative of $\Phi$ being conserved by the flow, rather than being some function of the vertical coordinate times a conserved quantity, as in pressure or log(pressure) coordinates.

In numerical atmospheric models, it is common to transform to $\sigma = p/p_s$, where $p_s(x, y, t)$ is the surface pressure, so that the lower boundary is a coordinate surface.

Another vertical coordinate that is especially useful when the flow is adiabatic ($d\Theta/dt = 0$) is $\Theta$ itself. Since it only makes sense to apply the hydrostatic equations when the flow is stably stratified, the transformation for *z* or *p* to $\Theta$ can be assumed to be well-defined. The simplification results from the fact that there is no "vertical" velocity with this coordinate, for adiabatic flow. The chain rule implies that

$$\begin{aligned} \nabla_h\Phi\big|_p &= \nabla_h\Phi\big|_\Theta - (\nabla_h p\big|_\Theta)\partial\Phi/\partial p = \nabla_h\Phi\big|_\Theta - (\nabla_h\Pi\big|_\Theta)\partial\Phi/\partial\Pi \\ &= \nabla_h(\Phi + c_p\Pi\theta)\big|_\Theta = \nabla_h M\big|_\Theta, \end{aligned} \tag{3.67}$$

where $M \equiv c_pT + gz$. *M* is the dry static energy, although in this context it is often referred to as the Montgomery streamfunction.

The equation for conservation of mass in isentropic coordinates can be obtained from the form of the mass element,

$$\delta m = -g^{-1}\delta x\delta y\delta p = -g^{-1}(\partial\Theta/\partial p)\delta x\delta y\delta\Theta, \tag{3.68}$$

which, assuming that $d\Theta/dt = 0$, leads to

$$\frac{d}{dt}(\partial\Theta/\partial p) = -(\partial\Theta/\partial p)\nabla_h \cdot \boldsymbol{V}_h. \tag{3.69}$$

An analogous transformation for a Boussinesq system, important in oceanography, is obtained by choosing density as the vertical coordinate. (See Problem 3.2)

Spherical geometry and rotation add some additional complications to discussions of the hydrostatic approximation. In order to retain the simple approximation that $\partial p/\partial r = -\rho g$ in the full vertical equation of motion, one must also ignore the vertical component of the Coriolis force, $2\Omega(\cos\theta)u$, and the metric term $(u^2 + v^2)/r$. Our scaling implies that the metric term is smaller than the pressure gradient by a factor of $H/a \approx \alpha(L/a)$. The ratio of the Coriolis to pressure-gradient forces in the vertical equation of motion is $f_0U/(U^2/H) = f_0H/U$. For $H = 10^4$ m, $f_0H \approx 1$ m/s. In the atmosphere, velocities and phase speeds of interest are typically an order of magnitude larger than this, and relevant values of *H* are substantially smaller as well, so one can justify neglect of this term empirically, even though it is not smaller than retained terms by a full aspect ratio.

Having omitted the Coriolis force from the vertical equation of motion, to retain energetic consistency the term $2\Omega(\cos\theta)w$ in the zonal momentum equation must also be discarded. The omission of the metric term in the vertical equation requires that the metric terms involving *w* in the zonal and meridional equations be omitted, for the same reason. If we now want to retain an angular momentum conservation law, the radial coordinate *r* in the remaining metric terms and in the material derivative must be replaced by a constant value, *a*. More generally, whenever the horizontal gradient operator $\nabla_h$ appears (in the pressure gradients, the substantial derivatives, and in the equation for conservation of mass), it must be computed as if the flow were confined to the surface of a sphere of radius *a*, ignoring variations in *r* within the atmosphere. This combination of assumptions is referred to as the *traditional approximation*, and leads to the following set of *primitive equations* for the atmosphere:

$$du/dt = 2\Omega(\sin\theta)v + uv(\tan\theta)/a - (\rho a(\cos\theta))^{-1}\partial p/\partial\lambda \tag{3.70}$$

$$dv/dt = -2\Omega(\sin\theta)u - u^2(\tan\theta)/a - (\rho a)^{-1}\partial p/\partial\theta \tag{3.71}$$

$$\partial p/\partial z = -\rho g \tag{3.72}$$

$$d\Theta/dt = 0 \tag{3.73}$$

$$d\rho/dt = -\rho\nabla\cdot\boldsymbol{V}, \tag{3.74}$$

where

$$d/dt = \partial/\partial t + (\cos\theta)^{-1}u\partial/\partial\lambda + a^{-1}v\partial/\partial\theta + w\partial/\partial z \tag{3.75}$$

and

$$\nabla\cdot\boldsymbol{V} = (a(\cos\theta))^{-1}\partial u/\partial\lambda + (a(\cos\theta))^{-1}\partial((\cos\theta)v)/\partial\theta + \partial w/\partial z. \tag{3.76}$$

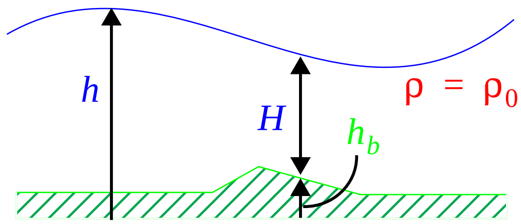
Exercise: These equations are the starting point for most global numerical atmospheric models used for weather prediction and climate studies. Confirm that they do possess natural energy and angular momentum conservation laws.

It is a plausible requirement of any model used for studies of climate that it possess exact analogs of the energy and angular momentum conservation laws. The manner in which this was assured in the derivation of the primitive equation model appears to be somewhat *ad hoc*. A natural way of insuring the existence of such laws is to start with a Hamiltonian rather than the equations of motion themselves. The Hamiltonian formulation of fluid dynamics is not without its subtleties, however, and we forego this approach here.

## 3.6 Shallow-water and layered models

The hydrostatic flow of a homogeneous (uniform density) incompressible fluid is of special interest, both because it provides a simple framework in which to investigate the interactions between rotation, buoyancy, and advection, and because it is directly relevant as a model of some atmospheric phenomena.

Let $h_b$ and $h$ be the height of the lower and upper boundaries of this homogeneous layer of fluid above a geopotential surface. The lower boundary is assumed to be rigid and the upper boundary is assumed to evolve so as to maintain a constant pressure (which we set equal to zero)




at the boundary. The thickness of the layer is $H - h - h_b$. Given hydrostatic balance, the pressureis simply the weight of the fluid above the point in question,

$$p(x, y, z) = g(h(x, y) - z), \tag{3.77}$$

where the uniform density has been set equal to unity. Therefore, the horizontal pressure gradient is $\nabla p = g\nabla h$. (All vectors are assumed to be horizontal, *i.e.*, to lie on a constant geopotential surface.) Since this pressure gradient is independent of height within the layer, we can satisfy the equation of motion by assuming that the flow is also independent of height. The resulting primitive equation of motion in spherical coordinates is

$$du/dt = fv + uv(\tan\theta)/a - g[a(\cos\theta)]^{-1}\partial h/\partial\lambda \tag{3.78}$$

$$dv/dt = -fu - u^2(\tan\theta)/a - ga^{-1}\partial h/\partial\theta, \tag{3.79}$$

while conservation of mass takes the form

$$dH/dt = -H\nabla\cdot\boldsymbol{V}. \tag{3.80}$$

The first term ensures that the 3-dimensional flow at $z = h_b$ has no component normal to the surface. Thus, the vertical velocity varies linearly from the value $\boldsymbol{V}\cdot\nabla h_b$ at the lower boundary to the value $d(H + h_b)/dt = dH/dt + \boldsymbol{V}\cdot\nabla h_b$ at the upper surface.

If we assume that the domain of interest is small compared to the radius of the earth, we can ignore the spherical geometry, including the metric terms, and set $f = f_0 = 2\Omega\sin\theta_0$, a constant. The governing equations become

$$du/dt = fv - g\partial h/\partial x \tag{3.81}$$

$$dv/dt = -fu - g\partial h/\partial y \tag{3.82}$$

$$dH/dt = -H(\partial u/\partial x + \partial v/\partial y), \tag{3.83}$$

with $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$. These are also the equations for a rotating laboratory shallow-water flow, except that $f = 2\Omega$ in the laboratory, since the rotation axis is parallel to the gravitational force.

**Problems**

3.1 Consider an ocean at rest with the vertical profiles of potential temperature and salinity $\Theta(z)$ and $S(z)$. Suppose that we know the equation of state in the form $\rho = \rho(\Theta, S, p)$. What is the expression for the buoyancy frequency?

3.2 In the set of equations 3.29 for linear waves in an isothermal atmosphere, set the vertical motion of the perturbation identically to zero, and solve the resulting equations to find the Lamb-wave solution. Confirm that the kinetic energy density of Lamb waves decreases with height.

3.3 If one uses density as a vertical coordinate in a Boussinesq fluid, what is the form of the equation of motion and the equation for conservation of mass?