A (brand spanking) new finite-volume dynamical core on the cubed-sphere using a fast Riemann solver

Xi Chen[1], Shian-Jiann Lin[2], and Lucas M. Harris[2]

[1]Program in Atmospheric and Oceanic Sciences, Princeton University, [2]NOAA/Geophysical Fluid Dynamics Laboratory, Princeton University, Princeton, New Jersey

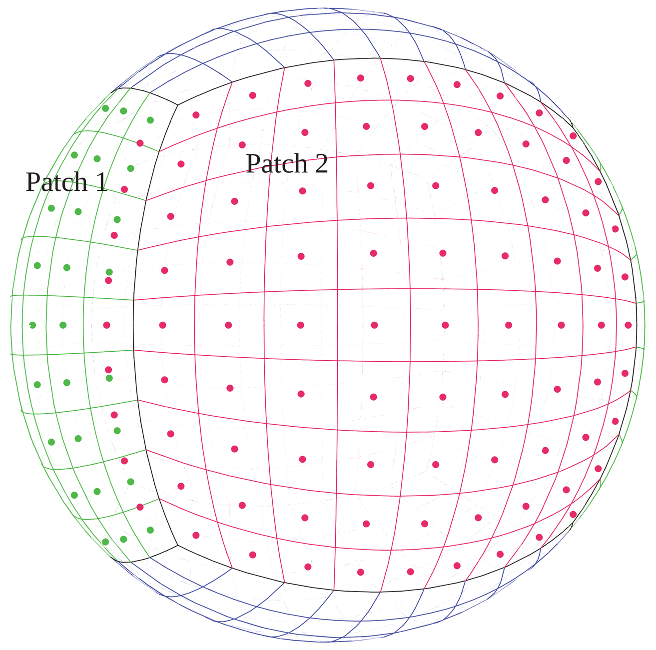
Abstract:

A cubed-sphere grid system provides much better uniform grid point distribution compared to the conventional lat-lon grid system. A finite volume method could be used to solve the flux form shallow water equations. However, due to the non-orthogonal cubed-sphere grid, extra metric terms are necessary in the momentum equations, which requires more computational steps compared to their counterpart on an orthogonal grid. In this model, the vector-invariant shallow water equations are implemented, which has fewer non-orthogonal terms. A fast Riemann solver is created to calculate the values of the prognostic variables at control volume interfaces for the momentum equations and to compute the fluxes for the continuity equation.

Theory:

The vector-invariant shallow water equations:

$$\frac{\partial h}{\partial t} = -\frac{1}{g}\left(\frac{\partial}{\partial x^1}(\sqrt{g}hu^1) + \frac{\partial}{\partial x^2}(\sqrt{g}hu^2)\right)$$
$$\frac{\partial u_1}{\partial t} = \sqrt{g}u^2(f+\zeta) - \frac{\partial}{\partial x^1}E$$
$$\frac{\partial u_2}{\partial t} = -\sqrt{g}u^1(f+\zeta) - \frac{\partial}{\partial x^2}E$$
$$E = G(h+h_s) + \frac{1}{2}(u_1u^1 + u_2u^2)$$
$$\zeta = \frac{1}{\sqrt{g}}\left(\frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2}\right)$$



Yang et al. (2010)

u_1,u_2 are the covariant components of the velocity. u^1, u^2 are the contravariant components of the velocity. A map space (x^1, x^2) represents a general curvilinear coordinate system.

Our finite volume method maintains the mass conservation, but does not maintain the momentum conservation. Each control volume uses a centered 1D 5-points stencil to calculate h and $\boldsymbol{v}$ at the volume interfaces. The velocity at control volume interface is converted to components that are perpendicular and parallel to the volume interfaces. h and $\boldsymbol{v}$ are mismatched between neighbor volumes at the interfaces, thus creates a "Riemann problem". A fast Riemann solver is created based on Chen et al. 2013 to calculate the volume interface values of h and $\boldsymbol{v}$ using the mismatched values at the volume interfaces:

$$\tilde{h}^* = \frac{1}{2}\left(h^L + h^R\right) + \frac{1}{2}\sqrt{\frac{h^L+h^R}{2G}}\left(u_\perp^L - u_\perp^R\right)$$
$$\tilde{u}_\perp = \frac{1}{2}\left(u_\perp^L + u_\perp^R\right) + \frac{1}{2}\sqrt{\frac{2G}{h^L+h^R}}\left(h^L - h^R\right)$$
$$\tilde{u}_{//} = \begin{cases} u_{//}^L & \text{if } \tilde{u}_\perp > 0 \\ u_{//}^R & \text{else} \end{cases}$$

Values of h and $\boldsymbol{v}$ at "mid-state" of the Riemann problem are simplified to constant as long as the speed of the gravity wave can be treated as constant at the volume interface. This approximation is valid because the speed of the gravity wave only appears in terms, which are used as implicit diffusion for the numerical scheme.

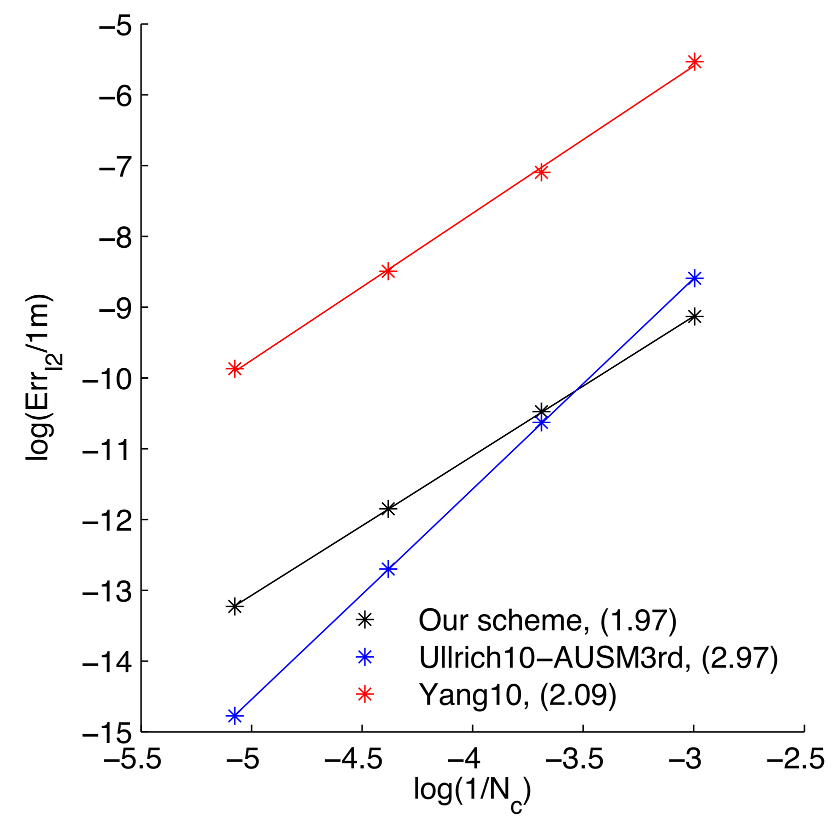
Equal-angular gnomonic cubed-sphere grid is used in our model, and the sphere is divided into six patches. Ghost cells is added to extend the patch boundaries for our finite volume approach. Since the grid center points of ghost cells and the corresponding cells in the neighbor patch lie on the same lines that parallel to the path interfaces, only 1D polynomial interpolation is necessary to remap the values of h and $\boldsymbol{v}$ for the ghost cells.

Numerical Results

The Williamson et al. (1992) test cases for shallow water models (W92) are used to benchmark our model.

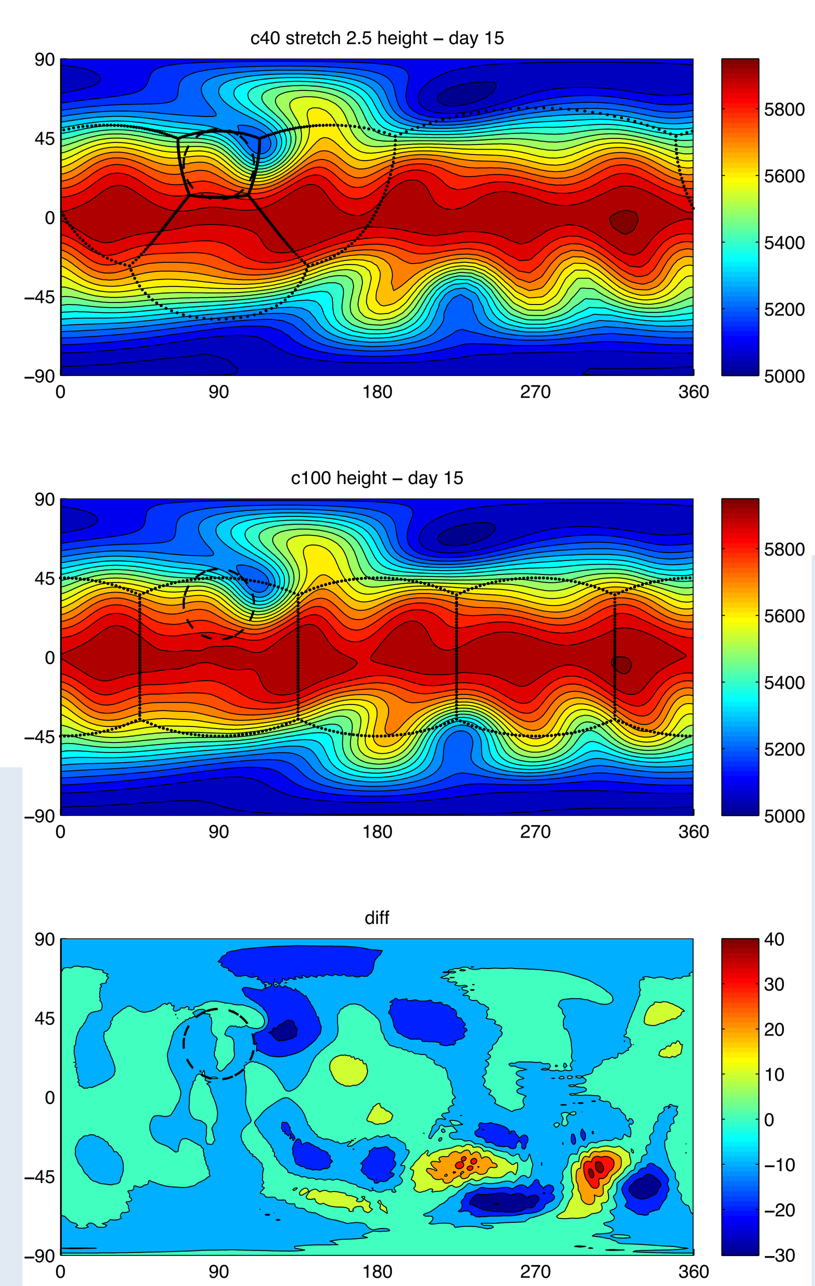
a. Steady Geostrophically Balanced Flow

W92 case 2 is a balanced geostrophic flow which should remain steady; deviations from steady state are errors. The results show a 2nd order convergence in our model. Although the 3rd order model developed by Ullrich et al. (2010) had a better convergence rate, our model shows lower L2 error at lower resolution and is more efficient. Our errors are also lower than those of the implicit method of Yang et al (2010), which allows larger time step than our explicit model but much more diffusive. Both Ullrich et al. and Yang et al. use the conservative momentum equations, which introduces more metric terms and thus more expense than our vector-invariant equations.



b. Zonal flow over an isolated mountain

This test is the test case 5 in W92. The initial status of winds and height is identical to the steady state geostrophically balanced flow in previous section, with zonal direction. However, the zonal flow impinges on a conical mountain. Our result shows good agreement with literatures. Note that with the low-diffusive Riemann solver, our results keeps good intensity of the flow height field perturbation. Our results are no more diffusive than other 3rd or 4th order numerical schemes including both FVM and SEM. The performance with the stretched grid is also good.



Height field of W92 case 5 with c40 stretched grid, c100, and difference Between the two results

Main References:

Chen, X., N. Andronova, B. Van Leer, J. E. Penner, J. P. Boyd, C. Jablonowski, and S.-J. Lin, 2013: A control-volume model of the compressible euler equations with a vertical lagrangian coordinate. Monthly Weather Review, 141 (7), 2526–2544

Ullrich, P. A., C. Jablonowski, and B. van Leer, 2010: Riemann-solver-based high-order finite-volume models for the shallow-water equations on the sphere. J. Comput. Phys., 229, 6104–6134.

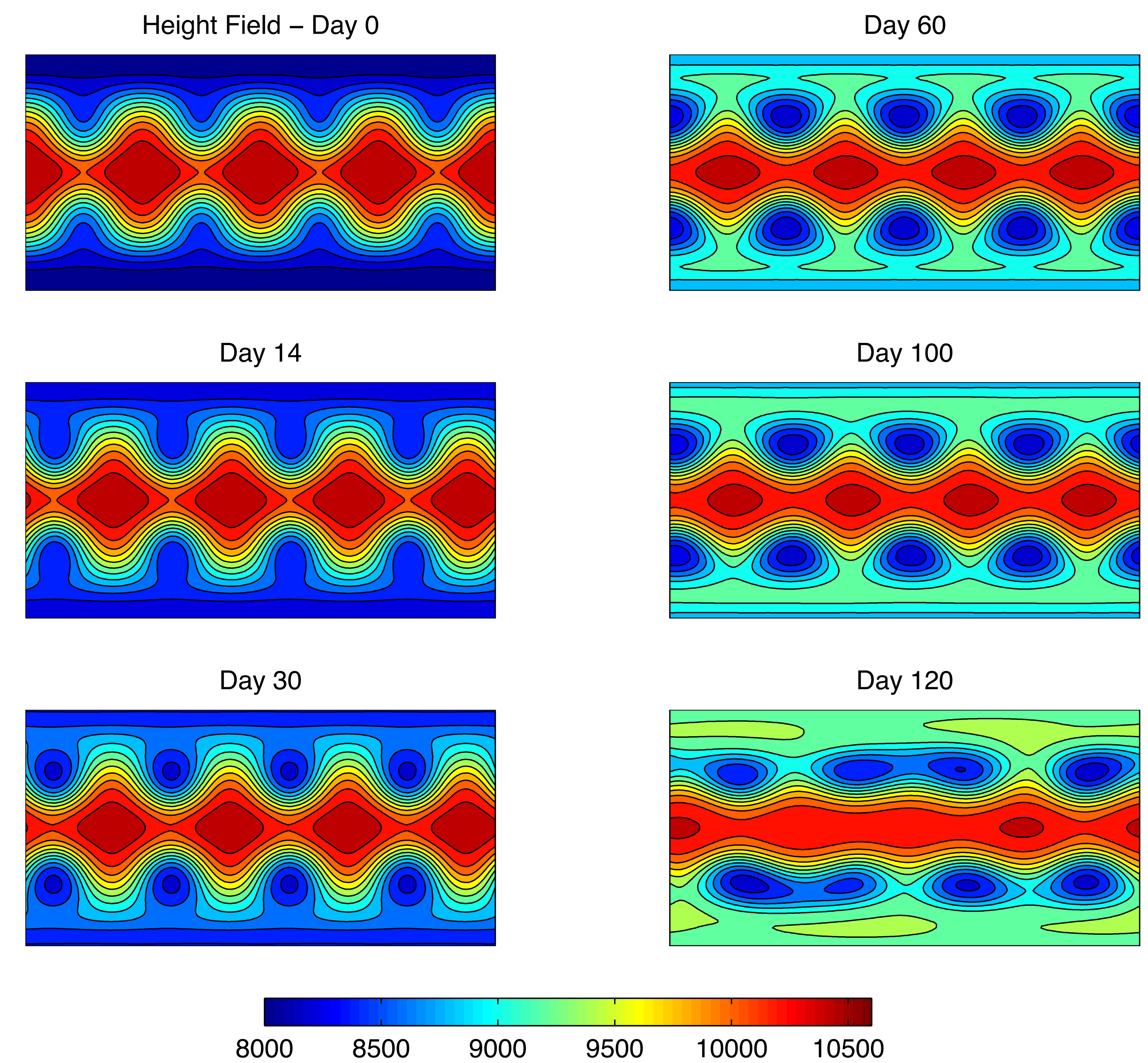
Williamson, D. L., J. B. Drake, J. J. Hack, R. Jakob, and P. N. Swarztrauber, 1992: A standard test set for numerical approximations to the shallow water equations in spherical geometry. Journal of Computational Physics, 102 (1), 211–224

Yang, C., J. Cao, and X. Cai, 2010: A fully implicit domain decomposition algorithm for shallow water equations on the cubed-sphere. SIAM Journal on Scientific Computing, 32 (1), 418–438

Contact: *xic@princeton.edu*

c. Rossby-Haurwitz wave

The Rossby-Haurwitz wave is an solution of the barotropic vorticity equation on the sphere. We performed the wave number 4 test because it is sensitive to instability due to the model's truncation error, and will eventually collapse into a unstructured turbulent flow. The time of the breakdown varies based on the numerical scheme employed. We used a c40 grid and the breakdown time is around day 100.



Height field of Rossby-Haurwitz wave with c40 grid resolution and wavenumber 4.

Conclusions

This shallow water model is designed for both accuracy and computational performance. We use the finite volume method with unstaggered grids on a cubed-sphere grid. A computationally efficient shallow water Riemann solver is created based on a fully compressible fluid Riemann solver developed by Chen et al. (2013). Our test shows that our model is 2nd order accurate with low diffusivity, and it handles nonlinear flow with good performance.

Next Step

This model shares the same set of prognostic variables with the current GFDL dynamical core, except the velocity is defined in A-grid. The governing equations are identical in both models. Chen et al 2013 demonstrated a a-grid Riemann solver based multi-layer fully compressible model using a vertical Lagrangian coordinate, which is similar to Lin 2004 approach. The numerical treatment in the vertical direction is isolated from the horizontal advection in such a vertical coordinate. Thus, this model is potentially possible to be extended to full 3D version by inheriting the vertical numerical treatment implemented in the current GFDL dynamical core.