ATMOSPHERIC DYNAMICS

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Contents

1 Preliminaries

1.1 Brief Review of Vector Calculus

Consider two points, $\boldsymbol{r}$ and $\boldsymbol{r} + \boldsymbol{\delta\ell}$, separated by an infinitesimal distance, where the components of the vector $\boldsymbol{\delta\ell}$ in Cartesian coordinates are $(\delta x, \delta y, \delta z)$: $\boldsymbol{\delta\ell} = \hat{\boldsymbol{x}}\delta x + \hat{\boldsymbol{y}}\delta y + \hat{\boldsymbol{z}}\delta z$. Consider also a scalar field ξ, which might for example, be the atmospheric temperature or density. Setting $\delta\xi = \xi(\boldsymbol{r} + \boldsymbol{\delta\ell}) - \xi(\boldsymbol{r})$, a linear approximation to this difference yields

$$\delta\xi = \frac{\partial\xi}{\partial x}\delta x + \frac{\partial\xi}{\partial y}\delta y + \frac{\partial\xi}{\partial z}\delta z\,. \tag{1.1}$$

The gradient $\boldsymbol{\nabla}\xi$ is defined so that

$$\delta\xi = \boldsymbol{\delta\ell} \boldsymbol{\cdot} \boldsymbol{\nabla}\xi, \tag{1.2}$$

that is,

$$\boldsymbol{\nabla}\xi = \frac{\partial\xi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\xi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\xi}{\partial z}\hat{\boldsymbol{z}}\,. \tag{1.3}$$

For any two points A and B and a path $\mathcal{P}$ connecting these points:

$$\xi(B) - \xi(A) = \int_{\mathcal{P}} \boldsymbol{\nabla}\xi \boldsymbol{\cdot} \boldsymbol{\delta\ell}\,. \tag{1.4}$$

The line integral over any closed path of the gradient of a scalar vanishes, as can be seen by setting $B = A$ in this expression.

The divergence of the vector field $\boldsymbol{F} = (F_x, F_y, F_z)$ in Cartesian coordinates is

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\,. \tag{1.5}$$

Gauss's theorem relates the integral of the divergence of $\boldsymbol{F}$ over a volume to the outward flux of $\boldsymbol{F}$:

$$\int\int\int_V \nabla \cdot \boldsymbol{F} dV = \int\int_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA\,. \tag{1.6}$$

Here $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the surface S bounding the volume V and directed from the interior to the exterior of the volume.

The curl of a vector field $\boldsymbol{A} = (A_x, A_y, A_z)$, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{A}$, is defined in Cartesian components, $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$, as

$$\begin{aligned} \omega_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\,, \\ \omega_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\,, \\ \omega_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\,. \end{aligned} \tag{1.7}$$

The curl of any gradient vanishes: $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\xi = 0$, as does the divergence of a curl: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})$. Stokes' theorem relates the curl to the line integral around a closed loop L:

$$\int_L \boldsymbol{A} \cdot \boldsymbol{\delta\ell} = \int\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} dS\,. \tag{1.8}$$

Here S is any surface whose boundary is the loop L and $\hat{\boldsymbol{n}}$ is an outwardly directed unit normal on that surface, with "outward" defined consistently with the orientation of the line integral around the loop, using the right-hand rule.

Vector identities related to the divergence and curl can be generated more easily by writing the curl in tensor notation.

$$(\boldsymbol{\nabla} \times \boldsymbol{A})_i = \sum_{j,k} \epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\partial_j A_k \tag{1.9}$$

Here $\partial_x \equiv \partial/\partial x$, each of the indices runs over (x, y, z), and the last expression uses the convention in which repeated indices are automatically summed over, The tensor ϵ_{ijk} is defined as equal to 0 if any of the two indices are equal, 1 if (i, j, k) is an even permutation of $(1, 2, 3)$ and -1 for odd permutations (i.e, $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$; $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$). The key identity, which can be confirmed by direct computation, is

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.10}$$

where δ_{ij} equals 1 if the two indices are equal and 0 otherwise. One can then show, for example, that

$$(\boldsymbol{\nabla} \times \boldsymbol{A}) \times \boldsymbol{A} = (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{A} - \frac{1}{2}\nabla|\boldsymbol{A}|^2 \tag{1.11}$$

by noting that

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{A})\times\boldsymbol{A})_i &= \epsilon_{ijk}\epsilon_{j\ell m}(\partial_\ell A_m)A_k \\ &= -\epsilon_{jik}\epsilon_{j\ell m}(\partial_\ell A_m)A_k \\ &= -(\delta_{i\ell}\delta_{km}-\delta_{im}\delta_{k\ell})(\partial_\ell A_m)A_k \\ &= -A_k\partial_i A_k + A_k\partial_k A_i\end{aligned} \tag{1.12}$$

Another useful identity that can be obtained with a similar manipulation is

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\nabla}\cdot\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{A}\,. \tag{1.13}$$

1.2 Eulerian and Lagrangian Perspectives

Let the vector field $\boldsymbol{v}(\boldsymbol{r},t) = (u,v,w)$ be the time-evolving velocity of a fluid: in the time δt each infinitesimal parcel of fluid is displaced by $\boldsymbol{v}\delta t$. The time derivative of a scalar $\xi(\boldsymbol{r},t)$ *moving with the flow* is

$$\frac{D\xi}{Dt} = \lim_{\delta t\to 0}\frac{\xi(\boldsymbol{r}+\boldsymbol{v}\delta t, t+\delta t) - \xi(\boldsymbol{r},t))}{\delta t} = \frac{\partial\xi}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\xi \tag{1.14}$$

Expressing this *material derivative* in terms of the cartesian components of the flow,

$$\frac{D\xi}{Dt} = \frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y} + w\frac{\partial\xi}{\partial z} \tag{1.15}$$

We can replace the scalar ξ in this expression by a vector, for example the velocity itself, so that the acceleration, the rate of change of the velocity following a particle of fluid, is

$$\frac{D\boldsymbol{v}}{Dt} = \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{\partial\boldsymbol{v}}{\partial t} + u\frac{\partial\boldsymbol{v}}{\partial x} + v\frac{\partial\boldsymbol{v}}{\partial y} + w\frac{\partial\boldsymbol{v}}{\partial z} \tag{1.16}$$

In Cartesian coordinates, the unit vectors are independent of space and time, so that

$$\frac{D\boldsymbol{v}}{Dt} = \hat{\boldsymbol{x}}\frac{Du}{Dt} + \hat{\boldsymbol{y}}\frac{Dv}{Dt} + \hat{\boldsymbol{z}}\frac{Dw}{Dt}. \tag{1.17}$$

The expression for, say, the x-component of the acceleration then takes the same form as does the material derivative of a scalar, simply replacing ξ by u in Eq. xx.

Rather than describe a fluid flow by thinking of the velocity field, and other fields of interest, as a function of space and time (the *Eulerian* description), one can instead, at least formally, label individual fluid particles and follow how fields evolve following these particles (the *Lagrangian* description). Particles can be labelled by their location at some time $t = 0$. The trajectory of a particle is obtained by solving

$$\frac{\partial \boldsymbol{x}(t,\boldsymbol{a})}{\partial t} = \boldsymbol{v(x,t)};\ \boldsymbol{x}(0,\boldsymbol{a}) = \boldsymbol{a} \tag{1.18}$$

The acceleration of a fluid particle is then simply the partial derivative of the velocity, holding the particle label fixed, but other interesting quantities, such as the pressure gradient force, become harder to express. We will focus entirely on Eulerian equations of motion, but a Lagrangian perspective, in which one thinks about particle trajectories and how fields of interest change along trajectories, will still be very important. Eq () has the form of a general three-dimensional nonlinear dynamical system,

$$\begin{aligned}\dot{x} &= u(x,y,z,t)\\ \dot{y} &= v(x,y,z,t)\\ \dot{z} &= w(x,y,z,t)\end{aligned}$$

so it should not be surprising that trajectories can be exceedingly complex, even chaotic, in what appear to be relatively simple velocity fields.

Particle trajectories can be difficult to visualize even in flows that are so simple that the trajectories are not chaotic. Consider, for example, a transverse wave in the presence of a uniform background flow $(u,v,w) = (U, v(x), 0)$ with $v(x) = V\sin(k(x-ct))$, with U, V constants. As particles in this flow move steadily towards positive x, their y-coordinate, η, varies as

$$\eta = \frac{V}{k(U-c)}\sin(k(x-ct)) \tag{1.19}$$

When the wave's phase speed c approaches the speed of the background flow U, the particle displacements due to the wave grow; the smaller $U-c$, the more time particles spend in one phase of the wave before entering the opposite phase.

This flow is non-divergent in the (x,y) plane, with $\partial_x u + \partial_y v = 0$, and can be described by a streamfunction ψ defined so that $(u,v) = (-\partial_y\psi, \partial_x\psi)$. Here $\psi = -Uy + (V/k)\sin(k(x-ct))$. In the special case that the streamfunction is independent of time, particles will flow along *streamlines*, the lines of constant ψ. A common trap that is difficult to avoid is to think that streamlines in a time-dependent flow resemble trajectories. In the example just described, the meanders in the streamlines due to the wave have amplitude V/kU, independent of the phase speed of the wave. A simple way of obtaining the trajectories is to move to a reference frame that is moving with the wave, so that the flow and its streamfunction are then independent of time.

A more realistic picture of what happens to trajectories when the background flow speed equals a wave phase speed is obtained by adding shear to the background flow, replacing the constant U with $u = U(y) = \Lambda y$. The resulting trajectories are illustrated in Fig. xx, as the reader may wish to confirm by computing the streamfunction in the frame moving with the wave. Near the location where $U(y) = c$ there are regions where particles are trapped by the wave and, on average, move with the wave phase speed. These regions have a width proportional to $(V/k\Lambda))^{1/2}$. On either side of these regions, particles are carried by the background flow. These critical layer or "cat's eye" structures will play a role in our discussion of wave-mean flow interaction in Ch. xx.

Another class of flows that results in counterintuitive trajectories is illustrated by setting $(u, v, w) = (A\sin(\omega t), 0, A\cos(\omega t))$. If A is a constant, the particle orbits are circles of radius A/ω in the $x - z$ plane, traversed counterclockwise. Now suppose that A is a function of z, but slowly varying in the sense that A varies by a small fraction of itself over the distance A/ω (that is, $\partial A/\partial z << \omega$). The result is that particles drift toward negative x, if A increases with z, with speed $(\partial A/\partial z)A/\omega$ (Fig. xx). The time average of the Eulerian flow is identically zero, but the Lagrangian mean flow of the fluid particles is non-zero, an effect referred to as Stokes' drift (owing to Stokes' classic analysis of this phenomenon in water waves). One can also create the opposite situation, in which fluid particles do not move systematically, yet the Eulerian flow has non-zero time mean – one can simply add a constant flow to balance the Stokes' drift. This latter situation may seem contrived, but it plays an important role in discussions of stratospheric dynamics (Ch. xx)

1.3 Line segments, Volumes, and Conservation of Mass

Consider the evolution of an infinitesimal line segment $\boldsymbol{\delta\ell}$ moving with the flow $\boldsymbol{v}$. The orientation and length of the vector $\boldsymbol{\delta\ell}$ change due to the difference in the velocity between the endpoints of the line segment:

$$\frac{D\boldsymbol{\delta\ell}}{Dt} = (\boldsymbol{\delta\ell} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{1.20}$$

In Cartesian coordinates, the rate of change of the x-component of $\boldsymbol{\delta\ell}$ is

$$\frac{D\delta x}{Dt} = \delta x\frac{\partial u}{\partial x} + \delta y\frac{\partial u}{\partial y} + \delta z\frac{\partial u}{\partial z} \tag{1.21}$$

The first term stretches the segment in the x-direction, while the second and third tilt the segment from the y- and z-directions into the x-direction.

Now follow an infinitesimal material parcel moving with the flow $\boldsymbol{v}$ and examine its volume δV as a function of time. The divergence of $\boldsymbol{v}$ is the fractional rate of change of this volume:

$$\frac{1}{\delta V}\frac{D\delta V}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{1.22}$$

Once can see this by choosing the small volume to be a cube at some initial time. For the line segment oriented along an edge parallel to the x-axis, we have, setting $\delta y = \delta z = 0$ in (**??**),

$$\frac{1}{\delta x}\frac{D\delta x}{Dt} = \frac{\partial u}{\partial x} \tag{1.23}$$

and

$$\frac{1}{\delta V}\frac{D\delta V}{Dt} = \frac{1}{\delta x}\frac{D\delta x}{Dt} + \frac{1}{\delta y}\frac{D\delta y}{Dt} + \frac{1}{\delta z}\frac{D\delta z}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{1.24}$$

The mass of an air parcel δM is the density ρ multiplied by the volume of the parcel δV. If the mass is conserved following the flow, then

$$\frac{D\delta M}{Dt} = \frac{D\rho\delta V}{Dt} = 0 \rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\delta V}\frac{D\delta V}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{1.25}$$

Conservation of mass can be expressed in this advective form

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{1.26}$$

or the flux form

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) \tag{1.27}$$

Given any scalar ξ that is conserved following the flow, except for the source/sink per unit mass S, we have in advective form

$$\frac{D\xi}{Dt} = S \tag{1.28}$$

or flux form

$$\frac{\partial\rho\xi}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\xi) = \rho S \tag{1.29}$$

The advective form of these expressions are naturally most useful when considering rates of change following fluid particles, while the flux forms are most useful when considering the rate of change at a fixed point in space.

For example, the diffusive flux of water vapor in the atmosphere is directed down the gradient of the water vapor mixing ratio, q, the ratio of the density of the water vapor to the density of the remaining (dry) air component. It is this mixing ratio that is conserved following a parcel in the absence of diffusion – the density of water vapor in the parcel will change if the volume of the paercel changes in the absence of diffusion, but the density of the dry air will change proportionally. If there are sources or sinks of water vapor due to evaporation or condensation totalling S, then the total mass of air is not conserved, but the mass of dry air is conserved. Setting the density of dry air equal to ρ_d, and the water vapor mixing ratio equal to q, we have $Dq/Dt = S$ and

$$\frac{\partial \rho_d q}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_d \boldsymbol{v} q) = \rho_d S \tag{1.30}$$

(The maximum value of the water vapor mixing ratio in the atmosphere is abour 2%, so the difference between ρ and ρ_d is often ignorable.)

1.4 Diffusion

Up to this point we have ignored molecular diffusion, which is always present, although it may be negligible for some contexts. Kinetic theory of gases leads to Fick's law, also empirically valid for liquids, that the diffusive flux is proportional to, and directed down the gradient of the tracer mixing ratio in question. Diffusion tends to equalize mixing ratios. The diffusive flux $\boldsymbol{F} = -\kappa \nabla q$ must be added to the advective flux:

$$\frac{\partial \rho \xi}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \xi) = \rho S - \nabla \cdot \boldsymbol{F} \tag{1.31}$$

1.5 Thermodynamics of an ideal gas

The equation of state for an ideal gas can be written as

$$p = \rho R T \tag{1.32}$$

where T is the absolute temperature, and R is the gas constant. Alternatively, $p = nkT$ where n is the number of molecules per unit volume, and k is Boltzmann's constant. (Using this notation, R is inversely proportional to the mean molecular weight.) For dry air, $R \approx 287(m^2/s^2)/K$.

On large scales the atmosphere is hydrostatic, with a balance in the vertical equation of motion between gravity and the pressure gradient:

$$\frac{\partial p}{\partial z} = -\rho g = -\frac{p}{H}\,;\; H \equiv \frac{RT}{g}, \tag{1.33}$$

with pressure, and density, decreasing exponentially in the vertical with scale height H for an isothermal atmosphere.

All motions that we are concerned with are *adiabatic* in the sense that the evolution is sufficiently slow that molecular collisions maintain local thermodynamic equilibrium within each (macroscopically) infinitesimal fluid parcel. The first law of thermodynamics can be written as

$$\frac{De}{Dt} = -p\frac{D\alpha}{Dt} + Q = -p\alpha\boldsymbol{\nabla}\cdot\boldsymbol{v} + Q \tag{1.34}$$

where e is the internal energy per unit mass, $\alpha \equiv 1/\rho$ and Q is the external heating. Alternatively, using the definition of the enthalpy per unit mass, $h = e + p\alpha$,

$$\frac{Dh}{Dt} = \alpha\frac{Dp}{Dt} + Q \tag{1.35}$$

In general, e and h are functions of pressure and temperature. The specific heats at constant volume (or density) c_v and constant pressure c_p are defined by

$$c_v \equiv \left.\frac{de}{dT}\right|_v;\;\; c_p \equiv \left.\frac{dh}{dT}\right|_p \tag{1.36}$$

so that $c_v DT/Dt = Q$ at constant volume and $c_p DT/Dt = Q$ at constant pressure. For an ideal gas, $h = e + RT$ and $c_p = c_v + R$.

Each degree of freedom excited at the temperatures of interest contributes $kT/2$ per molecule to the heat capacity at constant volume, or $RT/2$ per unit volume. The atmosphere consists mostly of diatomic molecules with 5 relevant degrees of freedom (3 translational and 2 rotational, with the vibrational degree of freedom unexcited by collisions at atmospheric temperatures), so that $c_v \approx 5R/2$ and $c_p \approx 7R/2$. These expressions turn out to be quite accurate for a dry atmosphere, resulting in a value of $c_p = 10^3 m^2/(s^2K)$. Common ratios of the heat capacities are given their own symbols:

$$\gamma \equiv \frac{c_p}{c_v} \approx 7/5;\;\; \kappa \equiv \frac{R}{c_p} \approx 2/7\,. \tag{1.37}$$

The mean surface pressure on Earth is roughly $10^5 Pa$. Dividing by the gravitational acceleration g yields a mass per unit area for the atmosphere of $10^4 Kg/m^2$. The heat capacity (at constant pressure) of an atmospheric column is then $10^7 (J/m^2)/K$, which is roughly the heat capacity of 2.4 meters of water.

The enthalpy equation is often written in the form

$$\frac{DT}{Dt} = \frac{\kappa T \omega}{p} + \frac{Q}{c_p} \tag{1.38}$$

where ω is the traditional notation for Dp/Dt. (Ambiguities arising from also using the symbol ω for vorticity, and frequency, are generally resolvable from the context). Except on the smallest scales, pressure is a monotonically decreasing function of z, and ω is referred to as the *pressure-coordinate vertical velocity.*

The entropy per unit mass of an ideal gas is defined so that

$$\frac{Ds}{Dt} = \frac{Q}{T} \tag{1.39}$$

From the enthalpy equation (), this requires

$$\frac{Ds}{Dt} = c_p \left(\frac{1}{T}\frac{DT}{Dt} - \frac{\kappa}{p}\frac{Dp}{Dt}\right) = c_p \frac{D\ln(Tp^{-\kappa})}{Dt} \tag{1.40}$$

It is convenient to define *potential temperature*, Θ, by the expression

$$s = c_p \ln \Theta \tag{1.41}$$

in terms of which (xx) is satisfied by setting

$$\Theta = T\left(\frac{p_*}{p}\right)^{\kappa} \tag{1.42}$$

where p_* is an arbitrary reference pressure, conventionally chosen to be $10^5 Pa$, close to the mean sea level pressure. The potential temperature is the temperature realized by adiabatic displacement to the pressure p_*.

For an atmosphere in hydrostatic balance, then

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{\kappa}{p}\frac{\partial p}{\partial z} = \frac{1}{T}\left(\frac{\partial T}{\partial z} - \frac{g}{c_p}\right) \tag{1.43}$$

If potential temperature (or entropy) is independent of height, the *lapse rate*, the rate of temperature decrease with height is the *adiabatic lapse rate*, $g/c_p \approx 9.8K/km$.

A familiar *parcel displacement* argument relates the atmospheric lapse rate to gravitational stability. Starting with an atmosphere in hydrostatic balance, lift a small parcel adiabatically by a distance δz, assuming that in doing so one does not perturb the pressure field (an assumption that can be justified), and compute the density of the lifted parcel. To the extent that parcel's density no longer equals that of the environment ρ_0, but differs by an amount $\delta\rho$, the pressure gradient no longer balances the gravitational acceleration, and instead

$$\frac{Dw}{Dt} = \frac{d^2}{dt^2}\delta z = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} = -g\frac{\delta\rho}{\rho_0} \tag{1.44}$$

But the density of the parcel, compared to that of the environment, given an unperturbed pressure field, is

$$\frac{\delta\rho}{\rho} = -\frac{\delta\Theta}{\Theta} = -\delta z\frac{\partial\Theta}{\partial z} \tag{1.45}$$

with the second equality following from the fact that Θ is conserved following the parcel. Therefore,

$$\frac{d^2}{dt^2}\delta z = -N^2\delta z \tag{1.46}$$

where

$$N^2 \equiv \frac{g}{\Theta}\frac{\partial\Theta}{\partial z} \tag{1.47}$$

N is referred to as the *buoyancy* or *Brunt-Vaisala frequency.* An atmosphere with a lapse rate smaller than the adiabatic lapse rate is gravitationally stable, N being a measure of this stability, while a *super-adiabatic* atmosphere in which the lapse rate is greater than the adiabatic value is unstable.

1.6 Equation of Motion

Ignoring viscous forces, the forces that we are typically concerned with in the fluid dynamics of the atmosphere are the force of gravity and the force generated by pressure gradients. Denoting the gravitational potential by Φ_G and the pressure field as p, we have for the inviscid equation of motion in an inertial frame of reference

$$\rho\frac{D\boldsymbol{v}}{Dt} = \rho(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Phi_G \tag{1.48}$$

The gravitational force exerted by the atmosphere on itself is negligible, so we can think of Φ_G as being specified independently of the state of the atmosphere. Additionally, we will not be considering the gravitational tides, so we can ignore the gravitational effects of the Sun and Moon and concern ourselves only with the gravitational field of the Earth. Using the identity (xx), one can rewrite this equation in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\omega} \times \boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}(\Phi_G + \frac{|v|^2}{2}) \tag{1.49}$$

For convenience, we typically move to a non-inertial coordinate system rotating with the constant angular velocity of the Earth. (The Earth's angular velocity actually varies on a variety of time scales, and some of these variations are of meteorological interest in that they are due to exchanges of angular momentum between the atmosphere and the solid Earth, but these variations are far too small to be of dynamical significance.) The rotating frame of reference adds two additional terms to the equations of motion: the centripetal acceleration, which can be written as the gradient of a potential and added to the gravitational potential, and the all-important Coriolis force.

The velocity $\boldsymbol{v}(\boldsymbol{r})$ of a fluid in solid body rotation with angular velocity $\boldsymbol{\Omega}$ is $\boldsymbol{\Omega} \times \boldsymbol{r}$, where $\boldsymbol{r}$ is vector pointing from the axis of rotation to the point in question, Therefore, the velocity fields in rotating and inertial reference frames are related by

$$\boldsymbol{v}_I(\boldsymbol{r}, t) = \boldsymbol{v}_R(\boldsymbol{r}, t) + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{1.50}$$

More generally, for any vector, the material derivatives in inertial and rotating frames are related by

$$\frac{D_I \boldsymbol{A}}{Dt} = \frac{D_R \boldsymbol{A}}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{A} \tag{1.51}$$

We then have for the acceleration

$$\begin{aligned} \frac{D_I \boldsymbol{v}_I}{Dt} &= \frac{D_R \boldsymbol{v}_I}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{v}_I \\ &= \frac{D_R \boldsymbol{v}_R}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{v}_I \\ &= \frac{D_R \boldsymbol{v}_R}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \end{aligned} \tag{1.52}$$

If $\boldsymbol{R}$ is defined to be the component of $\boldsymbol{r}$ perpendicular to $\boldsymbol{\Omega}$, we can write

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = -\Omega^2 \boldsymbol{R} = -\frac{1}{2}\Omega^2 \boldsymbol{\nabla} |\boldsymbol{R}|^2 \tag{1.53}$$

The resulting equation of motion is

$$\frac{D_R \boldsymbol{v}_R}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v}_R = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Phi \tag{1.54}$$

where

$$\Phi \equiv \Phi_G + \frac{1}{2}\Omega^2 |\boldsymbol{R}|^2 \,. \tag{1.55}$$

If the flow is at rest in an inertial frame, then the flow is in hydrostatic balance, with the pressure gradient in balance with the gravitational force. Since the gradients are then parallel, surfaces of constant pressure are also surfaces of constant gravitational potential, Φ_G. If, instead, the flow is at rest in the rotating coordinate system, then sufaces of constant pressure are also surfaces of constant *geopotential* Φ.

If the Earth were a perfect sphere of radius a with a radially symmetric gravitational potential, which we approximate near the surface as gz with z the height above the surface, then the surface pressure at the pole would be equal to the pressure at the equator at a height $z = \Omega^2 a^2/(2g)$, or about 20 km. Assuming a scale height of roughly 6 km, the atmosphere would then have a dramatic equatorial bulge, with the surface pressure at the equator a factor of 10 or so larger than the surface pressure at the poles. Needless to say, this bears no resemblance to the actual situation! The surface of the Earth is, in fact, close to being a surface of constant geopotential, with an equatorial bulge, and not a surface of constant gravitational potential. But this bulge is small compared to the radius of the Earth, and, in addition, we will always be working close enough to the surface that we can approximate surfaces of constant geopotential as spheres. With this understanding, we simply set the geopotential Φ equal to gz.

Using (xx), an alternative form of the equation of motion (xx) is

$$\frac{\partial \boldsymbol{v}_R}{\partial t} = -(\boldsymbol{\omega}_R + 2\boldsymbol{\Omega}) \times \boldsymbol{v}_R - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}(\Phi + \frac{|\boldsymbol{v}_R|^2}{2}) \tag{1.56}$$

The vorticity of solid body rotation is simply $2\boldsymbol{\Omega}$. This is most easily confirmed by using Cartesian coordinates, with the z-axis along the axis of rotation, so that $(u, v) = (-\Omega y, \Omega x)$ and then computing the z-component of the vorticity, $\partial_x v - \partial_y u$. The other components of the vorticity must vanish by symmetry. The vorticity as viewed in the inertial frame and in the rotating frame are related by

$$\boldsymbol{\omega}_I = \boldsymbol{\omega}_R + 2\boldsymbol{\Omega} \tag{1.57}$$

We refer to $\boldsymbol{\omega}_I$ as the *absolute* vorticity and $\boldsymbol{\omega}_R$ (usually denoted simply as $\boldsymbol{\omega}$), as the relative *vorticity.* Note how writing the RHS of (xx) in terms of the absolute vorticity subsumes the Coriolis force. As we will consistently use a rotating coordinate system in the following, we drop the subscript R in the following unless there is some chance of ambiguity.

1.7 Energetics

In the inviscid case, taking the dot product of the equation of motion with the velocity yields an equation for the rate of change of the kinetic energy density.

$$\rho \frac{D}{Dt} \frac{1}{2} |\boldsymbol{v}|^2 = -\rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \Phi - \boldsymbol{v} \cdot \boldsymbol{\nabla} p \tag{1.58}$$

or, for the rate of change of kinetic plus potential energy,

$$\rho \frac{D}{Dt} (\frac{1}{2} |\boldsymbol{v}|^2 + \Phi) = -\boldsymbol{\nabla} \cdot (p\boldsymbol{v}) + p \boldsymbol{\nabla} \cdot \boldsymbol{v} \tag{1.59}$$

While we have ignored viscosity above, when we consider the energy cycle in the atmosphere we need to keep in mind the dissipation of kinetic energy by viscosity.

In the atmosphere there is a tremendous disparity between the scales of motion that contain the bulk of the energy in the flow, on which most of our interest will be focused, and scales at which molecular diffusion and viscosity becomes significant. The time required to diffuse away a feature of scale L in the absence of flow is $L^2\mathcal{D}$, where $\mathcal{D}$ is the diffusivity , or viscosity in the case of momentum. The ratio of this diffusive time scale to the advective time L/U, is referred to as the Peclet number for tracers and the Reynolds number for momentum. The order of magnitude of kinematic molecular viscosities or diffusivities in the atmosphere near the surface is $10^{-5}m^2/s$. Large scale flows in the Earth's atmosphere typically have velocity scales of 10 m/s and vertical scales larger than 1 km, leading to Reynolds number of 10^9. A hardly perceptible puff of wind near the surface with a speed of 1 m/sec and a modest vertical scale of 10 m still has advection dominating viscosity by a factor of 10^6.

The hope and expectation is that a theory for the circulation of the atmosphere would not involve the value of the molecular viscosity in any significant way. This expectation is based on the analogy with fully developed three-dimensional turbulence at very high Reynolds numbers, for which it is found that the rate at which energy cascades to small scales detemines the rate of dissipation of energy, while the value of the viscosity simply determines the scale at which the dissipation takes place. In fact, this is more than an analogy, since all significant dissipation in the atmosphere (below heights of $\approx 100km$ at least), is presumed to occur in patches of fully developed turbulence. Much of this dissipation occurs in the *planetary boundary layer*, in which persistent turbulence is present near the surface, but a significant fraction occurs in intermittent turbulence in the free troposphere ("free" $\rightarrow$ "above the planetary boundary layer").

(Actually, there is another process besides turbulent cascades that, a bit surprisingly, dissipates a significant amount of kinetic energy in the atmosphere. Raindrops fall at a rate determined by a balance between viscous drag and gravity. Energy is dissipated by viscosity in this process.)

It is adequate for our purposes to use the expression for viscosity appropriate for an incompressible flow (flow speedsa re small compared to the speed fo sound, and the scales on which the viscosity acts are much smaller than the scale height of the atmosphere). The additional term needed in the equation of motion () is $D\boldsymbol{v}/Dt = ... + \nu\nabla^2\boldsymbol{v}$, where ν is the kinematic viscosity.

Etc.

1.8 Spherical Coordinates

It should come as no surprise that spherical coordinates are useful in studies of atmospheric dynamics. In addition to the radial distance from the origin r, we use the symbol θ for latitude, which ranges from $-\pi/2$ at the south pole to $\pi/2$ at the north pole, and λ for longitude, which ranges from 0 to 2π. The notation $\boldsymbol{v} = (u, v, w)$ is conventional for the (zonal, meridional, vertical) components of the velocity field, with u postive for eastward flow, v positive for northward flow, and w positive for a radially outward flow. In the meteorological literature, and in common discourse, a wind with $u > 0$ is referred to as westerly (from the west), a wind with $v > 0$ as southerly, and so forth. A note of caution: oceanographers speak of eastward, rather than westerly, currents. Oceanographers want to know where currents are taking their ship, while meteorologists as interested in where the air is coming from.

It is helpful to be comfortable with the spherical coordinate expressions for the gradient, divergence, and curl of a flow. We can write the expression for an infinitesimal line segment in spherical coordinates as

$$\boldsymbol{\delta\ell} = \hat{\boldsymbol{\lambda}}(r\cos(\theta)\delta\lambda) + \hat{\boldsymbol{\theta}}(r\delta\theta) + \hat{\boldsymbol{r}}\delta r \tag{1.60}$$

For a scalar field ξ, in order to retrieve $\delta\xi = \boldsymbol{\delta\ell}\cdot\boldsymbol{\nabla}\xi$ the gradient must take the form

$$\boldsymbol{\nabla}\xi = \frac{1}{r\cos(\theta)}\frac{\partial\xi}{\partial\lambda}\hat{\boldsymbol{\lambda}} + \frac{1}{r}\frac{\partial\xi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\xi}{\partial r}\hat{\boldsymbol{r}} \tag{1.61}$$

and for the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos(\theta)}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial r} \tag{1.62}$$

Care must be taken when considering the material derivative of a vector in spherical coordinates, since the unit vectors are position dependent. For example, the material derivative of the zonal component of the velocity, Du/Dt, is not equal to the zonal component of the acceleration of a fluid particle. Rather, with some effort one can show that the components of the acceleration are

$$\begin{aligned}
\frac{D\boldsymbol{v}}{Dt}\cdot\hat{\boldsymbol{\lambda}} &= \frac{Du}{Dt} - \frac{uv\tan(\theta)}{r} + \frac{uw}{r} \\
\frac{D\boldsymbol{v}}{Dt}\cdot\hat{\boldsymbol{\theta}} &= \frac{Dv}{Dt} + \frac{u^2\tan(\theta)}{r} + \frac{vw}{r} \\
\frac{D\boldsymbol{v}}{Dt}\cdot\hat{\boldsymbol{r}} &= \frac{Dw}{Dt} - \frac{u^2+v^2}{r}
\end{aligned} \tag{1.63}$$

where the additional terms on the right are referred to as metric terms.

One can remember the form of these terms by considering the angular momentum and kinetic energy equations. The form of the metric terms in the zonal (east-west) equation of motion can be obtained by considering the material derivative of the angular momentum per unit mass, $M = ur\cos(\theta)$. (Here and in the following, the term *angular momentum* will always refer to the component of the angular momentum vector along the axis of rotation.)

$$\begin{aligned}
\frac{DM}{Dt} &= r\cos(\theta)\frac{Du}{Dt} + u\cos(\theta)\frac{Dr}{Dt} - ur\sin(\theta)\frac{D\theta}{Dt} \\
&= r\cos(\theta)\frac{Du}{Dt} + u\cos(\theta)w - ur\sin(\theta)\frac{v}{r} \\
&= r\cos(\theta)(\frac{Du}{Dt} - \frac{uv\tan(\theta)}{r} + \frac{uw}{r})
\end{aligned} \tag{1.64}$$

The fact that the metric terms must cancel when forming the rate of change of kinetic energy can then be used as a mnemonic for the remaining terms

$$\begin{aligned}
\frac{1}{2}\frac{D|\boldsymbol{v}|^2}{Dt} &= \frac{1}{2}\frac{D(u^2+v^2+w^2)}{Dt} \\
&= u(\frac{Du}{Dt} - \frac{uv\tan(\theta)}{r} + \frac{uw}{r}) \\
&+ v(\frac{Dv}{Dt} + \frac{u^2\tan(\theta)}{r} + \frac{vw}{r}) \\
&+ w(\frac{Dw}{Dt} - \frac{u^2+v^2}{r})
\end{aligned} \tag{1.65}$$

Applying Gauss's theorem to the infinitesimal volume element (with volume $r^2\cos(\theta)\delta r\delta\theta\delta\lambda$) oriented along the coordinate axes, one finds that the divergence must be defined as

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{1}{r\cos(\theta)}\frac{\partial u}{\partial \lambda} + \frac{1}{r\cos(\theta)}\frac{\partial(v\cos(\theta))}{\partial \theta} + \frac{1}{r^2}\frac{\partial(r^2 w)}{\partial r} \tag{1.66}$$

Applying Stokes' theorem to an infinitesimal loop oriented along a pair of coordinate axes, one can derive the expression for the component of the vorticity normal to this loop. In particular, using a loop on the surface of a sphere, we obtain for the radial component of the vorticity

$$\omega_r = \frac{1}{r\cos(\theta)}\frac{\partial v}{\partial \lambda} - \frac{1}{r\cos(\theta)}\frac{\partial(u\cos(\theta))}{\partial \theta} \tag{1.67}$$

We have already seen that the vorticity of solid body rotation with angular velocity $\boldsymbol{\Omega}$ is $2\boldsymbol{\Omega}$. The radial component of the vorticity of solid body rotation is encountered so often that we denote it by the symbol f, also referred to as the *Coriolis parameter*:

$$f \equiv 2\boldsymbol{\Omega} \cdot \hat{\boldsymbol{r}} = 2\Omega\sin(\theta) \tag{1.68}$$

One can verify this expression from the spherical coordinate form of the vorticity (xx) by setting $(u, v) = (\Omega r\cos(\theta), 0)$. The radial component of the vorticity of solid body rotation is a monotonic function of latitude, ranging from$-\pi/2$ at the south pole to $\pi/2$ at the north pole. We typically denote the radial component of the relative vorticity as ζ, so that the radial component of the absolute vorticity is $f + \zeta$.

Computing the components of the Coriolis force in spherical coordinates, we finally have for the equations of motion

$$\begin{aligned}
\frac{Du}{Dt} - \frac{uv\tan(\theta)}{r} + \frac{uw}{r} &= 2\Omega\sin(\theta)v - 2\Omega\cos(\theta)w - \frac{1}{\rho r\cos(\theta)}\frac{\partial p}{\partial \lambda} \\
\frac{Dv}{Dt} + \frac{u^2\tan(\theta)}{r} + \frac{vw}{r} &= -2\Omega\sin(\theta)u - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} &= 2\Omega\cos(\theta)u - \frac{1}{\rho}\frac{\partial p}{\partial r} - g
\end{aligned}$$

with D/Dt defined by (xx).

The form of the Coriolis force in the zonal equation can be appreciated once again by considering the rate of change of angular momentum. In solid body rotation, the angular momentum is the eastward wind $\Omega r\cos(\theta)$ times the moment arm $r\cos(\theta)$. The total angular momentum, expressed in terms of the flow in the rotating frame, is then

$$M = r\cos(\theta)(u + \Omega r\cos(\theta)) = r\cos(\theta)u + \Omega r^2\cos^2(\theta) \tag{1.69}$$

The material derivative of the first term has already been computed in (xx), leading to the metric terms in the zonal equation of motion. Computing the derivative of the solid body term

$$\begin{aligned} \frac{D}{Dt}\Omega r^2 \cos^2(\theta) &= (\frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial r})\Omega r^2 \cos^2(\theta) \\ &= r\cos(\theta)(-2\Omega \sin(\theta)v + 2\Omega \cos(\theta)w) \end{aligned}$$

Combined with (xx), we see that the zonal equation of motion in the rotating frame can be written

$$\frac{DM}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} \tag{1.70}$$

The Coriolis force in the zonal equation emerges from the advection of the angular momentum of solid body rotation. Since the Coriolis force is perpendicular to the velocity, the components of the Coriolis force in the meridional and vertical equations are such as to cancel the terms in the zonal equation when computing the rate of change of kinetic energy, just as for the metric terms.

Consider the special case of a steady ($\partial/\partial t = 0$) axisymmetric ($\partial/\partial \lambda = 0$) flow. The angular momentum is then conserved following the flow in the meridional-vertical plane. In particular, suppose that in one branch of this circulation there is meridional motion at fixed radial coordinate, $r = a$. Then $\Omega a^2 \cos^2(\theta) + ua\cos(\theta)$ is independent of latitude along this branch. Suppose this flow starts at the equator with flow at rest with respect to the surface of the Earth, $u(\theta = 0) = 0$. Then $u(\theta) = u_M(\theta)$, where

$$u_M \equiv \Omega a \frac{\sin^2(\theta)}{\cos(\theta)} \tag{1.71}$$

With a equal to the mean radius of the Earth (a), $\Omega a \approx 460m/s$, so if a circulation of this sort exits from the equator to 30 latitude, a rather substantial flow of $\approx 130m/s$ is generated. We can talk interchangeably of this flow as being created by the conservation of angular momentum as a ring of air moves polewards and, therefore, closer to the axis of rotation, or of the zonal flow as being accelerated as the poleward flow is turned to the right (in the Northern Hemisphere) by the Coriolis force.

Note that a flow in solid body rotation has a latitudinal gradient in its angular momentum, with the angular momentum decreasing polewards. In addition, for an arbitrary zonal flow $(u, v) = (u(\theta), 0)$, the radial component of vorticity is proportional to the latitudinal angular momentum gradient:

$$\frac{1}{r}\frac{\partial M}{\partial \theta} = -r\cos(\theta)(f + \zeta) \tag{1.72}$$

In the angular momentum conserving zonal flow with $u = u_M$, the radial component of the vorticity vanishes.

The *Rossby number* is a measure of the magnitude of the radial component of the relative vorticity as compared to the radial component of the vorticity of solid body rotation. If U is a characteristic magnitude of the horizontal flow and L a characteristic length scale for variations in this flow, then

$$Ro \sim \frac{|\zeta|}{f} \sim \frac{U}{fL} \tag{1.73}$$

One can think of the Rossby number as one possible non-dimensional measure of the departure of the flow from solid body rotation, focusing on the resulting distortion of the radial component of the vorticity.

1.9 Vorticity in Homogeneous Incompressible Flow

The equation of motion tells us how the flow will evolve, given the density and pressure distributions. At a minimum, one also needs equations for the evolution of the pressure and density in order to evolve the state of the atmosphere forward in time. Conservation of mass provides an equation for the evolution of the density. To evolve the pressure distribution in the case of an ideal gas, one can, for example, first relate the pressure to the density and temperature (or entropy) through the equation of state, and then compute the evolution of temperature (or entropy) from a heat equation. We will need to turn to this general situation, but we first consider a much simpler case of an incompressible flow with uniform density. Our interest in uniform density, incompressible flow is primarily in the special case of two-dimensional flow on the surface of a sphere.

An *incompressible* flow is one in which the divergence is assumed to vanish everywhere and at all times: $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. Scaling arguments suggest that incompressibility is a good approximation for an atmosphere when 1) the Mach number, the ratio of the flow velocity to the speed of sound, is small and 2) when the vertical scale of the flow is small compared to the scale height of the atmosphere over which the density varies by O(1) in the Earth's gravitational field. We will discuss these scaling arguments in (xx).

A *homogeneous* flow is one in which the density is uniform in space and time. (Inhomogeneous incompressible flows will occupy our attention in due course.) In the homogeneous, incompressible, and inviscid case, by taking the divergence of the equation of motion (xx) one obtains a Poisson equation for the pressure of the form

$$\nabla^2 p = -\rho \boldsymbol{\nabla} \cdot ((\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \nabla\Phi) \tag{1.74}$$

where the right hand side at time t can be computed from knowledge of the flow field at time t. With appropriate boundary conditions on the pressure, one can then think of the incompressibility condition as determining the pressure, and the evolution of the state of the system $(\boldsymbol{v}, p)$ is then fully defined. In situations of this sort, we will refer to the flow evolution as being determined by a *prognostic* equation, with an explicit time derivative, with the pressure being determined by a *diagnostic* equation or a *balance* condition, or as being *slaved* to the flow field.

For an incompressible flow, we have that

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{1.75}$$

This follows from (xx), using the incompressibility condition and the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$. Taking the curl of the equation of motion in the form (xx) and using this result, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -(\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) + ((\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{1.76}$$

or

$$\frac{D(\boldsymbol{\omega} + 2\boldsymbol{\Omega})}{Dt} = ((\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{1.77}$$

Comparing to Eq. (xx), we see that the absolute vorticity in a homogeneous, incompressible flow satisfies the same equation as does an infinitesimal material line segment. If a line segment is initially aligned along the absolute vorticity vector, it will stay aligned as the flow evolves, with the magnitude of the vorticity increasing or decreasing as the length of the line segment increases or decreases. Because of this property, we often speak of the vorticity as being *tilted* or *stretched*.

Suppose the existence of a scalar field that is conserved following the flow, $D\xi/Dt = 0$. If a conserved scalar is not readily available, one can be created by painting the flow with a continuously varying color. We then have

$$\frac{D}{Dt}[(\boldsymbol{\delta\ell} \cdot \boldsymbol{\nabla})\xi] = 0 \tag{1.78}$$

since the quantity in brackets is simply the difference in the value of ξ between the two endpoints of the material line segment, and these two values are separately conserved as the segement moves with the flow. But the vorticity and the material line segment satisfy the same equation, so we have also proven, for homogeneous incompressible flow, that

$$\frac{D}{Dt}[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \boldsymbol{\nabla}\xi] = 0. \tag{1.79}$$

Since this proof is a bit indirect, the reader may want to verify this equation directly by working out the derivatives using the vorticity equation and the equation for the conservation of ξ.

The quantity conserved here is referred to as the *potential vorticity* (*PV* for short). Changes in the gradient of the conserved scalar cancel the effects of the stretching and tilting of the vorticity, creating another conserved scalar from the vorticity vector.

Another centrally important perspective on vorticity evolution is provided by Kelvin's circulation theorem. This celebrated theorem states that the circulation around a material loop $\mathcal{C}$ is conserved for (inviscid) homogeneous incompressible flow:

$$\frac{D}{Dt}\int_{\mathcal{C}} \boldsymbol{v}\cdot\boldsymbol{\delta\ell} = 0 \tag{1.80}$$

We can prove this result by expanding the line integral into two terms (replace the integral by a discrete sum if you find this notation confusing)

$$\frac{D}{Dt}\int_{\mathcal{C}} \boldsymbol{v}\cdot\boldsymbol{\delta\ell} = \int_{\mathcal{C}} \frac{D\boldsymbol{v}}{Dt}\cdot\boldsymbol{\delta\ell} + \int_{\mathcal{C}} \boldsymbol{v}\cdot\frac{D\boldsymbol{\delta\ell}}{Dt} \tag{1.81}$$

The first term is zero because it is the integral of a gradient, from the equation of motion, since the density is assumed constant. (The assumption of uniform density is crucial here.) The second term vanishes because (xx) implies that it is also the integral of a gradient:

$$\boldsymbol{v}\cdot\frac{D\boldsymbol{\delta\ell}}{Dt} = \boldsymbol{v}\cdot[(\boldsymbol{\delta\ell}\cdot\boldsymbol{\nabla})\boldsymbol{v}] = \boldsymbol{\delta\ell}\cdot\boldsymbol{\nabla}(|\boldsymbol{v}|^2/2) \tag{1.82}$$

Applied to an infinitesimal material loop, Kelvin's theorem and Stokes theorem together imply that the component of the vorticity perpendicular to the loop, multiplied by the area of the loop $\delta\mathcal{A}$, is conserved:

$$\frac{D}{Dt}[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\hat{\boldsymbol{n}}\delta\mathcal{A}] = 0 \tag{1.83}$$

We can rederive potential vorticity conservation from this result by assuming that the loop lies in a surface of constant ξ, the conserved scalar. We then have the normal to the loop proportional to the gradient of ξ:

$$0 = \frac{D}{Dt}[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\frac{\boldsymbol{\nabla}\xi}{|\boldsymbol{\nabla}\xi|}\delta\mathcal{A}] = \frac{D}{Dt}[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\boldsymbol{\nabla}\xi\frac{h\delta\mathcal{A}}{\delta\xi}] \tag{1.84}$$

where h is the distance between two surfaces of constant ξ differing by $\delta\xi$. But $h\delta\mathcal{A}$ is the volume of the cylinder depicted in Fig xx, and is conserved following the flow since we have assumed incompressibility. By construction $\delta\xi$ is also conserved. We thereby recover xx.

1.10 Planar vs. Spherical Two-Dimensional Flow

Consider first the special case of inviscid two dimensional flow on a plane, in an inertial reference frame, with identically zero vertical velocity. For the three-dimensional flow to be incompressible, the flow must be non-divergent on the plane, and can be represented by a streamfunction $\psi(x, y)$:

$$\boldsymbol{v} = (u, v, 0) = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0) \tag{1.85}$$

The vorticity has only a vertical component in this special case:

$$\boldsymbol{\omega} = (0, 0, \zeta); \;\; \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{1.86}$$

It follows from the vorticity equation that the vertical component of the vorticity is conserved following the flow in this 2D case, there being no stretching or twisting:

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} = -J(\psi, \zeta) \tag{1.87}$$

where we use the shorthand Jacobian notation

$$J(A, B) \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial y} \tag{1.88}$$

We can rederive the result that the vertical component of vorticity is conserved following the flow from Kelvin's Circulation Theorem by simply choosing an infintesimal loop lying in the plane and noting that the area of the loop is conserved since the flow is non-divergence in the plane. Or one can use potential vorticity conservation by choosing the scalar to be the vertical coordinate z, which is conserved following the flow because the vertical velocity is zero by assumption.

Conservation of the vertical component of vorticity (or simply "vorticity") forms the basis of the fascinating theory of two-dimensional turbulence. What happens to the dynamics of this flow if we add rotation about the z-axis? (By *adding rotation* we mean that ψ is now the streamfunction of a two-dimensional flow in a rotating reference frame.) Kelvin's circulation or PV conservation immediately yields the result that Eq (xx) is completely unchanged, since the vertical component of the vorticity of solid body rotation is a constant.

Now consider the case of 2-dimensional flow on the surface of a sphere of radius a. (It may be useful at times to think of the fluid as having a finite but negligible thickness.) The flow is now assumed to be non-divergent on the sphere:

$$\frac{1}{a\cos(\theta)}\frac{\partial u}{\partial \lambda}+\frac{1}{a\cos(\theta)}\frac{\partial(v\cos(\theta))}{\partial\theta}=0 \tag{1.89}$$

which we can satisfy once again by defining a streamfunction

$$(u,v)=(-\frac{1}{a}\frac{\partial\psi}{\partial\theta},\frac{1}{a\cos(\theta)}\frac{\partial\psi}{\partial\lambda}) \tag{1.90}$$

The radial component of the vorticity is

$$\zeta = \frac{1}{a\cos(\theta)}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos(\theta)}\frac{\partial(u\cos(\theta))}{\partial\theta}=\nabla^2\psi \tag{1.91}$$

$$= \frac{1}{a^2\cos^2(\theta)}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{1}{a^2\cos(\theta)}\frac{\partial}{\partial\theta}(\cos(\theta)\frac{\partial\psi}{\partial\theta}) \tag{1.92}$$

Once again we use Kelvin's Theorem, now using a loop lying on the surface of our sphere. The result is that the *radial component of the vorticity is conserved following the flow.* Equivalently, we can use potential vorticity conservation, with the radial coordinate as the conserved scalar, to obtain the same result. So the evolution equation for the radial component of vorticity becomes simply

$$\frac{\partial\zeta}{\partial t}=-\frac{u}{a\cos(\theta)}\frac{\partial\zeta}{\partial\lambda}-\frac{v}{a}\frac{\partial\zeta}{\partial\theta} \tag{1.93}$$

All terms in this equation can be written in terms of the streamfunction. The vorticity can be inverted uniquely to obtain the streamfunction by solving Poissons equation (xx), which has a unique solution on the sphere.

For this 2D flow on a sphere, the vorticity field contains all information about the dynamical state. To evolve the flow forward in time, given the vorticity at some time t, we use conservation of vorticity to obtain the vorticity at $t+dt$, after which we compute the flow consistent with this new vorticity field (the flow *induced* by the vorticity distribution) by solving Poissons equation for the streamfunction at $t+dt$.

Zonal averages, averages around a latitude circle, are of special interest in theories of the atmospheric circulation. We denote the zonal average by an overbar:

$$\overline{u}\equiv\frac{1}{2\pi}\int_0^{2\pi}u d\lambda=\frac{C}{2\pi r\cos(\theta)} \tag{1.94}$$

Here C is the circulation around the latitude circle. Applying Stokes Theorem to this latitude circle, we find that the zonally averaged zonal wind at any latitude is proportional to the radial component of vorticity integrated over the polar cap bounded by this latitude circle. Rather than think about the forces which modify the zonal mean zonal wind, we can, if we prefer, think about the ways in which the radial component of vorticity integrated over this polar cap can change.

A fundamental distinction between the spherical and planar cases results when we add rotation. In this spherical case, the conserved quantity, the radial component of the vorticity, is not uniform in solid body rotation, but rather, is a monotonic function of latitude. In a rotating system we have

$$\frac{D(f+\zeta)}{Dt} = 0 \implies \frac{\partial \zeta}{\partial t} = -\frac{u}{a\cos(\theta)}\frac{\partial \zeta}{\partial \lambda} - \frac{v}{a}\frac{\partial (f+\zeta)}{\partial \theta} \tag{1.95}$$

or

$$\frac{D\zeta}{Dt} = -\beta v \tag{1.96}$$

where the meridional gradient of the radial component of the vorticity of solid body rotation is denoted by β:

$$\beta \equiv \frac{1}{a}\frac{\partial f}{\partial \theta} = \frac{2\Omega\cos(\theta)}{a} \tag{1.97}$$

The importance of the background vorticity gradient is apparent when one linearizes the dynamics about a state of rest in the rotating frame. When we *linearize* equations, we typically start with a solution of the equations (in this case simply $\boldsymbol{v} = 0$), and ask how small disturbances from this state evolve. Being small, we assume that terms quadratic in the disturbance can be ignored. Denoting the perturbation with a prime, we find that the the perturbations evolve according to

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\frac{\beta}{a\cos(\theta)}\frac{\partial \psi}{\partial \lambda} = -\frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} \tag{1.98}$$

By decomposing the initial condition into spherical harmonics (eignfunctions of the Laplacian on the sphere), one can write the general solution to this equation as a linear superposition of modal solutions (*Rossby-Haurwitz* waves):

$$\sum_{\ell=1,\infty}\sum_{m=-\ell,+\ell} A_{\ell m} P_{\ell m}(\cos(\theta))\exp(i(m\lambda - \omega_{\ell m} t)) \tag{1.99}$$

with

$$\omega_{\ell m} = -\frac{2m\Omega}{\ell(\ell+1)} \tag{1.100}$$

Here $P_{\ell m}$ are associated Legendre polynomials. All of these modes propagate westward. (The disturbance that propagates most rapidly westward

($\ell = m = 1$), is stationary in the inertial frame). This westward propagation is clearly a consequence of the background vorticity gradient.

This background monotonic vorticity distribution provides large-scale atmospheric and oceanic dynamics with much of its distinctive flavor, and it can be important even when our interest is confined to a small enough span of latitudes that the spherical geometry is otherwise irrelevant. For this reason, we frequently make use of the tangent-plane, or β-plane, approximation to spherical two-dimensional flow, which simply amounts to solving (xx) in Cartesian coordinates and assuming that β is a constant:

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial x} - \beta \frac{\partial \psi}{\partial x} = -J(\psi, \zeta + \beta y) \tag{1.101}$$

Linearizing this equation yields

$$\frac{\partial \zeta'}{\partial t} = -\beta v' \tag{1.102}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2}\right) = -\beta \frac{\partial \psi'}{\partial x} \tag{1.103}$$

Assuming a plane wave of the form

$$\psi' = \Re[A e^{i(kx + \ell y - \omega t)}] \tag{1.104}$$

results in the familiar *Rossby wave* dispersion relation

$$\omega = -\frac{\beta k}{k^2 + \ell^2} \tag{1.105}$$

At fixed y, lines of constant phase propagate westward with the speed $c = \omega/k = -\beta/(k^2 + \ell^2)$. Longer waves, with smaller wavenumbers, propagate faster. The background vorticity gradient is the source of the anisotropic propagation; if the sign of the vorticity gradient were reversed, the propagation would be eastward.

The local, tangent-plane approximation makes it easier to see the essence of the dynamics resulting in Rossby waves, isolating this local dispersion relation. Global modes, such as the Rossby-Haurwitz waves, or their counterparts in more realistic models, do have a role to play in meteorology but local Rossby wave-like dynamics is a dominant feature of the atmosphere. One can add walls at northern and southern boundaries to create a finite domain and mimic the spectrum of global modes obtained on the sphere, and this reentrant channel geometry is quite common for this and other reasons. But the walls create artificial features as well, unlike anything in the atmosphere, and one also commonly considers a doubly-periodic (or toric) geometry for the β-plane, in which the streamfunction in (xx) is assumed to be periodic in both x and y, especially in studies of two-diomensional turbulence.

2 Momentum fluxes in a barotropic model

2.1 Vorticity Mixing

We continue to consider a homogeneous incompressible flow on the surface of a sphere. Suppose that the fluid is initially in solid body rotation, with angular velocity Ω. (From now on, in this context of two dimensional flow on the sphere, the term vorticity will refer to the radial component of the vorticity vector.)

Suppose that the fluid is now stirred by some external agent (see Figure xx). We constrain the stirrer to act only within a well-defined range of latitudes. Outside of this directly stirred region vorticity is conserved following the flow. This is not true within the stirred region, since the imposed external stirring changes the vorticity on fluid particles.

Depending on the details of its motion, one can imagine that the stirrer might impart some horizontally integrated angular momentum to the fluid, pushing the entire spherical shell either eastward or westward on average. In fact, the appropriate physical picture of this stirring of the upper troposphere as due to vortex stretching associated with baroclinic eddy production (getting ahead of ourselves) necessarily involves the vertical transfer of angular momentum from the upper to the lower troposphere. From the perspective of a model of the upper troposphere, stirring with this realistic character would drain horizontally integrated angular momentum out of the layer as a whole. We ignore this point for the time being as we temporarily focus on the horizontal redistribution of angular momentum. Irrespective of the dynamics of the vertical redistribution, it is the divergence of the horizontal fluxes, when vertically integrated, that must balance the surface stresses generated by the near-surface winds.

Once the stirring begins, the fluid will transmit the disturbance to other latitudes (Fig. xx). Concentrating on a latitude circle outside of the directly stirred region, we see that the air that has moved northwards through this latitude has less vorticity than the air that has moved southwards, so there is a net flux of vorticity southwards, down the mean vorticity gradient, as the disturbance grows. The vorticity integrated over the polar cap bounded by this latitude circle has decreased. By Stokes' theorem, this implies that the zonally averaged zonal flow, $\overline{u}$, has decreased at this latitude.

The zonal mean flow is now easterly in a reference frame moving with the initial solid body rotation. This argument can be made for any latitude outside of the stirred region, on both the poleward and equatorward sides, as long as the disturbance reaches this latitude. (It makes no difference if one chooses to consider the latitude circle as the boundary of a polar cap extending to the South Pole, rather than the North Pole.) Since they can only redistribute angular momentum, the horizontal angular momentum fluxes generated by the stirring must be directed into the stirred region!

This explanation for acceleration ($\partial\overline{u}/\partial t > 0$) in the stirred region is a bit indirect – one first argues for deceleration ($\partial\overline{u}/\partial t < 0$) in the unstirred regions that become populated by eddies spreading outwards from the stirring; one then argues that the horizontal fluxes causing this deceleration must be accelerating the stirred region. One way of making the argument more direct is to assume that the stirring is concentrated in a short burst that quickly creates eddies, which is then followed by a quiescent period in which the stirring is turned off. During this latter period, vorticity is assumed to be conserved following fluid particles throughout the flow. To the extent that the disturbance spreads meridionally, eddy amplitudes can be expected to decay in that part of the flow that was originally perturbed. By a process opposite to that pictured in Figure 2, a *poleward* vorticity flux will be generated in this region, to the extent that the flow reverts to a more zonally symmetric form, and the zonal mean flow will thereby be accelerated in the stirred region as the eddies decay. The compensating deceleration occurs in those regions into which the disturbance spreads.

Suppose that the stirring is now turned off. If flow at some latitude outside of the stirred region in Figure xx relaxes back to a zonal flow with the vorticity on each particle being conserved, then the vorticity fluxes will be reversed, and $\overline{u}$ will return to its initial value. If we want a stirring pulse to generate a net deceleration at this latitude, some irreversible mixing must take place. To the extent that eddy vorticity is mixed into the environment, the vorticity transported during the decay phase will not fully compensate the transport during the growth stage. The rectified effect of an event in which some mixing occurs, in which pieces of vorticity break off and are dispersed into the environmental vorticity distribution, will be a net southward flux of vorticity and, therefore, deceleration of the flow in the unstirred regions.

A bit of imagination is required to see the connection between this picture and the Earths climate. In the context of this homogeneous non-divergent model, the claim is that one should think of the upper troposphere as being continually stirred in midlatitudes – by the vortex stretching created by baroclinic eddy production, a process that we will examine in due course The eddies generated by this stirring spread meridionally and mix vorticity irreversibly, and thereby transport angular momentum so as to accelerate the flow within the stirred region, and decelerate it to the north and south. In a statistically steady state, torques are required to balance these accelerations. The momentum flux convergence integrated through the depth of the atmosphere must be balanced by torques at the surface. Independent of the detailed manner in which angular momentum is transported vertically from the upper troposphere to the surface, we require mean surface westerlies in midlatitudes to provide surface torques to balance the acceleration due to the eddies in the stirred region, and surface easterlies to the north and south.

It is difficult to develop an intuitive understanding of this acceleration due to stirring if one focuses on the momentum or angular momentum fluxes. Following a fluid particle, momentum is modified by pressure gradients. In contrast, vorticity is conserved following the flow in our simple 2-dimensional homogeneous incompressible flow on a spherical surface. Because it is (we hope!) relatively easy to develop intuition about the fluxes of such materially conserved quantities, we presume that vorticity fluxes are inherently simpler to think about than are momentum fluxes.

The sign of the mean vorticity gradient is crucial to this argument, as it determines whether mixing the fluid will result in deceleration or acceleration. It is not necessary for the model atmosphere to be initially in solid body rotation for the argument to work as stated; it is only necessary that the northward vorticity gradient be positive. But this gradient will eventually be destroyed in a region where the eddies are successful in mixing the fluid, if there are no restoring forces acting. The deceleration in the mixed region will then cease. Therefore, a restoring force for the vorticity gradient, in the upper tropospheric regions where this mixing is occurring, would be an essential part of this barotropic picture if we were to try to use is as a basis for a consistent statistically steady model. But flow restoration is a part of the problem that is difficult to talk about realistically without generalizing to the inhomogenous baroclinic case, in which radiative forcing can restore potential vorticity gradients.

2.2 Eddy Vorticity and Momentum Fluxes

We have seen how a simple vorticity mixing argument, combined with a picture in which the upper troposphere is preferentially stirred in mid-latitudes, provides an intuitively appealing explanation for the zonal mean surface wind field. Before generalizing to a baroclinic atmosphere, we pause here to examine some of the concepts that emerge naturally as one attempts to describe more quantitatively how and why eddies modify the zonal mean flow in the atmosphere, including the key concepts related to pseudomomentum conservation. These concepts have a general applicability, but it can be difficult to develop an intuitive understanding of them if they are first encountered in the most general settings. We continue to work in the context of the simple two-dimensional, non-divergent model for the time being.

A simple manipulation, using the definition of vorticity, makes the relation between momentum fluxes and vorticity fluxes more explicit. Let us agree to ignore spherical geometry for the moment and consider a two-dimensional non-divergent flow on a plane, with x increasing eastwards and y increasing polewards. From the definitions of vorticity, ζ and divergence (which is assumed to vanish identically) we have

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= \zeta \end{aligned}$$

It follows directly that

$$v\zeta = \frac{1}{2}\frac{\partial}{\partial x}(v^2 - u^2) - \frac{\partial}{\partial y}(uv) \tag{2.1}$$

Let an overbar denote a zonal average, an average over x, where we assume that the flow is periodic in the x-direction, just as on the sphere. A prime denotes the deviation from this average. There can be no zonally averaged meridional flow in this non-divergent model, so the total zonally averaged northward flux of vorticity is given by the eddy flux, which, in turn, equals the convergence of the northward eddy flux of eastward momentum:

$$\overline{v\zeta} = \overline{v'\zeta'} = -\frac{\partial}{\partial y}\overline{u'v'} \tag{2.2}$$

A similar relationship exists on the sphere between the vorticity flux and the convergence of the angular momentum flux:

$$\overline{v'\zeta'} = -\frac{1}{a\cos^2(\theta)}\frac{\partial}{\partial \theta}(\cos^2(\theta)\overline{u'v'}) \tag{2.3}$$

In the $\cos^2(\theta)$ term on the right hand side, one factor of $\cos(\theta)$ takes into account the moment arm needed to convert momentum to angular momentum, while the other factor is present due to the convergence of the meridians. The latitudinal patterns of the zonal mean vorticity flux and angular momentum flux contain the same information; we can generate one from the other by differentiating or integrating. A southwards, negative, vorticity flux, as would be produced by mixing of vorticity down the gradient produced by planetary rotation, is equivalent to a divergence of the eddy angular momentum flux.

One can stare at this derivation for quite a long time and still not develop any intuition for why it exists. Yet we already have an intuitive understanding of this relationship from the discussion in xx. A southward flux of vorticity across some latitude circle removes vorticity from the polar cap north of this latitude, which, by Stokes Theorem, implies a reduction in the zonally averaged zonal flow. Assuming non-divergent flow on the surface of the sphere, fluid motions can change the zonal mean flow only through the convergence of eddy angular momentum fluxes, so the vorticity flux must be proportional to this convergence:

$$\frac{\partial \overline{u}}{\partial t} = -\frac{1}{a\cos(\theta)}\frac{\partial}{\partial \theta}(\cos^2(\theta)\overline{u'v'}) = \overline{v'\zeta'} \tag{2.4}$$

A seemingly innocuous but centrally important consequence of this relation is that the northward vorticity flux vanishes when integrated over latitude:

$$\int \overline{v'\zeta'}\cos^2(\theta)d\theta = 0 \tag{2.5}$$

The manner in which vorticity can be transported is evidently strongly constrained. In particular, if it is not identically zero, the integrand in this expression, the northward eddy vorticity flux, must be positive in some regions and negative in others. This constraint holds whether the flow is unforced and invisicid or forced and dissipative.

We evidently have to be careful when we try to think of vorticity as being mixed downgradient. In a complex turbulent 2-dimensional flow one might intuitively expect a preexisting gradient of vorticity to be mixed away through downgradient fluxes. This intuition presumably would rely on the fact that we expect the vorticity, like a passive tracer that is conserved following the flow, to be cascaded to small scales, eventually destroying large-scale gradients. But if the mean absolute vorticity distribution is initially monotonic, as is the case if we are close to solid body rotation on the surface of a sphere, one cannot mix vorticity downgradient everywhere, for then the eddy vorticity flux would be everywhere of one sign. It is not primarily our intuition about the downgradient turbulent mixing of vorticity that we have to re-examine, but rather the possibility of the existence of self-generated turbulence in the first place, when the flow is dominated by a monotonic vorticity distribution. We can always stir the flow externally to create eddies, but, as we have seen, our intuition that vorticity will be mixed must then be altered to take into account the fact that the stirring changes the vorticity on fluid particles.

A barotropic flow with a monotonic absolute vorticity distribution evidently has a certain stability. In the context of linear theory this is a well-known result, reviewed in the next section, associated with the names of Rayleigh and Kuo, but one need not restrict oneself to linear theory when thinking about the implications of this constraint, as first emphasized by Arnold. In contrast, if the mean vorticity gradient does change sign, then the fluid has the potential to spontaneously mix vorticity downgradient, through a process we refer to as barotropic eddy production.

2.3 Pseudomomentum

To create a more explicit statistically steady model, it is helpful to explicitly linearize the equations about an arbitrary zonal flow, $\overline{u(y)}$. The mean meridional, north-south, flow in in this non-divergent model is identically zero, and an arbitarary zonal flow is an exact solution of the invisicid equation of motion. Our interest is in the evolution of small perturbatons away from this state. We will assume a Cartesian model on a β-plane here for simplicity, but the reader can verify that all of the results below can be transferred to the sphere.

If we want to model a statistically steady state, we also need a way of dissipating eddy vorticity perturbations. To generate the simplest possible model, we simply damp the eddy vorticity uniformly. This damping will be the source of irreversibility in this linear system, playing the role dominated by irreversible wave breaking in more realistic models of the upper troposphere.

Once linearized, the advection of vorticity reduces to the sum of two terms, the zonal advection by the prescribed zonal flow of the perturbation vorticity and the meridional advection by the perturbation meridional flow of the basic state vorticity. The zonal flow $\overline{u}$ makes a contribution $\overline{\zeta} = -\partial\overline{u}/\partial y$ to the vorticity. Therefore, we can write the basic state meridional vorticity gradient as

$$\gamma \equiv \beta + \frac{\partial \overline{\zeta}}{\partial y} = \beta - \frac{\partial^2 \overline{u}}{\partial y^2} \tag{2.6}$$

The dynamics of these linear disturbances is then governed by the equation

$$\frac{\partial \zeta'}{\partial t} = -\overline{u}\frac{\partial \zeta'}{\partial x} - \gamma v' + s - \kappa \zeta' \tag{2.7}$$

here s is the stirring, κ is the inverse of the damping time. We assume for the time being that γ is positive at all latitudes.

Multiplying both sides of this equation by the perturbation vorticity and then averaging over x, we obtain an equation for the mean square eddy vorticity, or eddy *enstrophy*.

$$\frac{1}{2}\frac{\partial \overline{\zeta'^2}}{\partial t} = -\gamma \overline{v'\zeta'} + \overline{s'\zeta'} - \kappa \overline{\zeta'^2} \tag{2.8}$$

Eddy enstrophy, in addition to being created and destroyed by stirring and damping, is generated by a downgradient (southward in this case) eddy vorticity flux. This is an interesting equation, but it becomes even more interesting if we divide both sides by the mean vorticity gradient γ, and replace the vorticity flux by the momentum flux convergence:

$$\frac{1}{2\gamma}\frac{\partial \overline{\zeta'^2}}{\partial t} = -\frac{\partial}{\partial y}(-\overline{u'v'}) + \frac{1}{\gamma}\overline{s'\zeta'} - \frac{\kappa}{\gamma}\overline{\zeta'^2} \tag{2.9}$$

We now have an equation that has the form of a conservation law for a quantity $\mathcal{P}$ with flux $\mathcal{F}$ and source/sink Q:

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\partial \mathcal{F}}{\partial y} + Q \tag{2.10}$$

The conserved quantity is

$$\mathcal{P} = \frac{1}{2\gamma}\overline{\zeta'^2} \tag{2.11}$$

We refer to $\mathcal{P}$ as the density of *pseudomomentum*. The integral of $\mathcal{P}$ over latitude is conserved in time, if there is no stirring nor damping, and if there is no flux of $\mathcal{P}$ out of the domain.

On the sphere, one speaks of the pseudo-angular momentum (or is it the angular pseudomomentum?), which has an extra factor of $\cos(\theta)$ multiplying $\overline{\zeta^2}/(2\gamma)$, so that, in the absence of soources and sinks, and taking into account the convergence of meridians, the quantity

$$\int_{-\pi/2}^{\pi/2} \frac{\overline{\zeta^2}}{2\gamma}\cos^2(\theta)d\theta \tag{2.12}$$

is conserved for linear disturbances.

Conservation of momentum can be understood as a consequence of translational invariance of a physical system; pseudomomentum conservation equations of this type can be shown to be consequences of the translational invariance of the basic state on which the waves propagate. The flux of pseudomomentum, $\mathcal{F}$, is often referred to as the *Eliassen-Palm flux*. In this case, the flux of pseudomomentum is the negative of the momentum flux,

$$\mathcal{F} = -\overline{u'v'} \tag{2.13}$$

The source/sink for pseudomomentum consists of two terms $Q = S - D$, associated with the stirring and damping respectively. The reader might benefit from deriving the analogous results on the sphere.

We now have for the mean flow modification due to the waves,

$$\frac{\partial \overline{u}}{\partial t} = \overline{v'\zeta'} = -\frac{\partial \mathcal{P}}{\partial t} + S - D \tag{2.14}$$

Mean flow modification due to waves, in problems analogous to this two-dimensional example, is a consequence of wave transients (the rate of change of pseudomomentum), wave sources (S), and wave dissipation (D). This mean flow modification, being quadratic in the amplitude of the wave motions, can be ignored initially in the computation of the RHS. If this mean flow modification is sustained and no other forcing of the mean flow comes into play that constrains the evolution, then the mean flow change will eventually affect the eddy dynamics.

Consider the special case in which there are no wave sources or sinks, so that $S = D = 0$. Once again suppose that initially the disturbance is tightly localized in latitude, but as time evolves it spreads meridionally. As we have already seen, we expect that the mean zonal flow will increase in the region initially occupied by the eddy, and will decrease in those regions into which the eddy spreads. Expressing the eddy amplitudes in terms of the pseudomomentum allows us to determine by how much the mean flow is modified: the change in $\overline{u}$ is precisely equal in amplitude, and opposite in sign, to the change in $\mathcal{P}$.

For the case of an arbitrary linear disturbance, with no sources or sinks, the total pseduomomentum, the integral of the pseudomomentum density over the domain, is conserved. If the vorticity gradient γ is everywhere positive, this density is positive definite. As a result, if a disturbance is initially infinitesimal, it must always remain infinitesimal in order to conserve pseudomomentum; therefore the flow is stable.

There are various formal definitions of stability; the type we are referrring to here is often called stability in the sense of Liapunov, in which one bounds the growth of a quadratic, positive defiinite measure of the size of the disturbance. It captures the essense of our intuitive notion of stability, that an $O(\epsilon)$ disturbance cannot grow to be $O(1)$. Stability in this sense potentially depends on the way in which we measure size. Rather than pseudomomentum, suppose that we measure the size of a disturbance by integrating its enstrophy over the domain. Eddy enstrophy is not conserved, but it is easily checked that we have stability in the sense of enstrophy also. Eddy enstrophy can grow somewhat by moving from a latitude of small γ to one of larger γ, but the amount of growth possible is bounded as long as γ is bounded.

One can understand why the pseudomomentum takes the the form that it does by the following argument. Suppose the flow is unperturbed at time $t = 0$ in the neighborhood of some latitude $y = 0$. As some later time t, a perturbation has propagated into this region. Locate the initial positions of the particles that now find themselves at $y = 0$ at time t, and define $\eta(x : t)$ to be the y-coordinate of the curve defined by these initial points. The difference between the initial and final zonal mean flows is determined by the amount of vorticity that has passed through $y = 0$ in the intervening time. Since vorticity is conserved, this integrated transport is

$$\overline{u(t)} - \overline{u(0)} = L^{-1} \int_0^{\eta(x;t)} \zeta_I(y,0)dxdy \approx \frac{1}{2}\gamma\overline{\eta^2} \tag{2.15}$$

where L is the length of the channel, and the final expression on the RHS is obtaining by assuming, for small displacements, that we can approximate $\zeta_I \approx C + \gamma y$ and also set $\overline{\eta} = 0$.

$$\mathcal{P} = \frac{1}{2}\gamma\overline{\eta'^2} \tag{2.16}$$

. We can interpret η' as the meridional particle displacement required to create the vorticity perturbation, etc.

If, instead, the mean vorticity gradient changes sign with latitude, then there is the potential for growth since the contributions to the pseudomomentum from different latitudes can cancel. This is the Rayleigh-Kuo criterion for barotropic instability: the mean vorticity gradient must change sign. Rayleigh considered the non-rotating plane-parallel case, for which this criterion reduces to the statement that the flow curvature, $\partial_{yy}\overline{u}$ must change sign. Kuo extended this result to the rotating sphere, or equivalently, the β-plane considered here. While the mathematics is the same, the result is profoundly different. In the non-rotating case, a flow stable by the Rayleigh criterion is exceptional: the simplest jet profile, say $\overline{u} = sech^2(y/L)$, will have not one but two inflection points at which $\partial_{yy}\overline{u} = 0$. But on a rotating sphere, due to the background monotonic vorticity distribution represented here by the positive constant β, stability is the rule rather than the exception if the jet is sufficently weak. Rotation on the sphere is stablizing to these barotropic flows because of the associated background monotonic vortiicty distribution.

Returning to the general case with stirrring and damping, but now assuming that the eddy field is in a statistically steady state, we have instead

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial F}{\partial y} = S - D \tag{2.17}$$

The mean flow will be accelerated by the eddies where the source of pseudomomentum S is greater than the sink D. In a true statistically steady state, this acceleration in the upper troposphere would be balanced by momentum transport from the upper to the lower troposphere, where it would, in turn, be balanced by surface stresses. The stress exerted by the atmosphere on the surface is then simply proportional to the upper tropospheric $S - D$. As long as the eddies spread in latitude, we expect $S - D > 0$, and surface westerlies, in the source region.

The simple case of a steady unforced, inviscid wave is of special interest, for then the vorticity flux is zero and the momentum flux is independent of latitude, independent of the structure of the mean flow $\overline{u}$ through which the wave propagates.

An alternative perspective on the horizontal redistribution of angular momentum, based on the properties of Rossby waves, is often encountered and can be very useful. It requires that the flow remain linear and that the meridional variation of the zonal mean flow be slow enough so as to allow one to define a local dispersion relation and a group velocity. One can then show that the Rossby waves propagating to the north from the source region generate a southward flux of eastward angular momentum, and that the waves propagating to the south generate a northward flux. The convergence of eastward momentum in the source region accelerates the flow, while the divergence in the regions into which the waves are propagating produces deceleration.

The meridional group velocity of a Rossby wave is

$$\frac{\partial \omega}{\partial \ell} = \frac{2\beta k \ell}{(k^2 + \ell^2)^2} \tag{2.18}$$

The wave propagating northward must have positive meridional group velocity, or $k\ell > 0$. But the eddy zonal and meridional velocities in the wave are

$$u = A\ell sin(kx + \ell y - \omega t) \tag{2.19}$$

$$v = -A\,k sin(kx + \ell y - \omega t) \tag{2.20}$$

so the momentum flux averaged over x is simply

$$\overline{uv} = -\frac{k\ell A^2}{2} \tag{2.21}$$

which is negative in this case. Equatorward of the source, on the other hand, we must choose $k\ell < 0$ to insure a southward group velocity, resulting in a positive momentum flux. Figure xx shows the pattern of tilted streamlines consistent with this choice of signs for the product $k\ell$. Superposition of this pattern onto the mean westerlies produces the familiar tilted trough structure of midlatitude eddies.

This linear wave perspective can be very useful, but it disguises the explanation for the sign of the mean flow acceleration, making it appear to depend on the details of the Rossby wave dispersion relation. We have seen that the pattern of acceleration and deceleration follows directly from vorticity conservation, without any restriction as to the linearity or nonlinearity of the flow. Indeed, the connection between these two explanations seems mysterious at first glance (even at second glance) and is worth pursuing.

Why does $F = -\overline{uv}$ turn out to be the meridional flux of P? We have already encountered one fact that helps us understand this: the meridional group velocity of a Rossby wave packet is opposite in sign to the eddy momentum flux, so it is of the same sign as F. This makes sense, because P is simply a particular measure of eddy amplitude. If we are looking at a well-defined wave packet moving meridionally, P must move with it, and the sign of the flux of P must reflect this fact. Exploring this point further, assume that there are no sources or sinks for simplicity and consider the case of a wave packet moving with a well-defined group velocity G. Avoiding the formal derivation, we expect to be able to write down that

$$\frac{\partial P}{\partial t} = -\frac{\partial\, GP}{\partial y} \tag{2.22}$$

If G is independent of y, this equation just expresses the fact that the pseudomomentum moves with the wave packet. We place G inside of the y-derivative to cover the case in which G varies with y, to insure that the total amount of pseudomomentum is conserved. For a wave packet, we therefore expect that $F = GP$.

We can confirm this expression for the special case of uniform flow, $U =$ *const* with $\gamma = \beta$. Consider once again a wave packet consisting of a carrier wave with zonal and meridional wavenumbers k and ℓ, and an envelope with streamfunction amplitude A. As described above, the momentum flux is $-k\ell A^2/2$ and the meridional group velocity of the wavepacket is $G = 2\beta k\ell/(k^2+\ell^2)^2$. Additionally, ignoring derivatives of the amplitude modulation as compared to derivatives of the phase of the wave, the vorticity is $\approx (-(k^2+\ell^2)A/2$, so the pseudomomentum density is

$$P \approx A^2 \frac{(k^2+\ell^2)^2}{4\gamma} \tag{2.23}$$

and, indeed, $F \approx GP$! Despite the fact that I have worked through this derivation numerous times, I never fail to be impressed by the away in which the various wavenumber factors cancel, as they must, to produce this simple result.

It takes some effort to get accustomed to the idea of using pseudomomentum as a measure of eddy amplitude, rather than a more familiar measure such as eddy kinetic energy. In some respects, eddy kinetic is actually a more complicated quantity than is the pseudomomentum. For example, eddy kinetic energy, unlike pseudomomentum, is not conserved for linear disturbances on a sheared flow, even in the absence of wave sources and sinks. If we multiply Eq () by U and integrate over latitude, we obtain an equation for the rate of change of the zonal kinetic energy. The negative of this expression is the rate of change of eddy kinetic energy, since the total energy of the flow is conserved as long as the flow is inviscid and unforced. One can write the result in a couple of different ways; continuing to ignore spherical geometry the result is:

$$\frac{\partial}{\partial t}\frac{\overline{u'^2+v'^2}}{2} = -\int \overline{u}\,\overline{v'\zeta'}dy = \int \frac{\partial}{\partial y}(\overline{u}\,\overline{u'v'})dy \tag{2.24}$$

Integration by parts has been used to obtain the last expression. The familiar consequence is that eddy kinetic energy is generated by an eddy momentum flux that is directed down the mean velocity gradient. On the sphere, it is the gradient of *angular velocity*, $U/cos(\theta)$ that is important in this regard, as the reader may wish to verify.

We therefore have the result that a wave packet propagating from strong westerlies into weaker westerlies will lose energy, for its momentum flux will be directed upgradient; conversely, a packet propagating from weak to strong westerlies will gain energy. When we see an eddy gaining energy from the mean flow we are often tempted to look for an instability, a temptation that should be resisted in the case of a disturbance propagating through a shear flow that is stable by the Rayleigh-Kuo criterion.

2.4 Eddy Energetics

An equation for the rate of change of the kinetic energy for this incompressible, two dimensional flow can be obtained by taking the dot product of the equation of motion with $\boldsymbol{v}$,

$$\frac{\partial}{\partial t}(\frac{1}{2}|\boldsymbol{v}|^2) = -\boldsymbol{\nabla}\cdot[\boldsymbol{v}(\frac{1}{2}|\boldsymbol{v}|^2 + \frac{p}{\rho})] \tag{2.25}$$

Kinetic energy changes locally becuse it is advected in from elsewhere and because for the work performed by the pressure force. Energy transfer due to pressure forces complicates atmospheric energetics. Averaging around a latitude circle, and considering the Cartesian β-plane for simplicity,

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{(u^2+v^2)} = -\frac{\partial}{\partial y}\overline{v(\frac{1}{2}(u^2+v^2)+\frac{p}{\rho})} \tag{2.26}$$

The total kinetic energy at any latitude can be decomposed into two parts: the energy in the zonal mean flow, $\mathcal{Z} = \overline{u}^2/2$, and the energy in the deviations from the zonal mean flow, $\mathcal{E} = \overline{(}u'^2 + v'^2)/2$, the latter referred to as the eddy kinetic energy. (Recall that $\overline{v} = 0$ in this incompressible 2D flow0. From xx we have

$$\frac{\partial \mathcal{Z}}{\partial t} = \overline{u}\overline{v'\zeta'} = -\overline{u}\frac{\partial}{\partial y}\overline{u'v'} \tag{2.27}$$

and subtracting from (xx) results in the eddy kinetic energy equation

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= \overline{u}\frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial y}\overline{v'(\frac{1}{2}(u'^2+v'^2)+\overline{u}u'+\frac{p'}{\rho})} && (2.28)\\ &= -\overline{u'v'}\frac{\partial\overline{u}}{\partial y} - \frac{\partial}{\partial y}\overline{v'(\frac{1}{2}(u'^2+v'^2)+\frac{p'}{\rho})} && (2.29)\end{aligned}$$

For small eddies, we retain only the quadratic terms on the RHS,

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= \overline{u}\frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial y}(\overline{u}\overline{u'v'} + \frac{1}{\rho}\overline{v'p'}) && (2.30)\\ &= -\overline{u'v'}\frac{\partial\overline{u}}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial y}\overline{v'p'} && (2.31)\end{aligned}$$

so that, integrating over the domain (assuming no fluxes at the boundaries of the domain)

$$\int \mathcal{E}dy = -\int \overline{u'v'}\frac{\partial\overline{u}}{\partial y}dy = -\int \mathcal{Z}dy \tag{2.32}$$

Eddy energy grows if the eddy momentum fluxes are downgradient on average. Unlike pseudomomentum, eddy kinetic energy is not conserved for linear waves on a shear flow. Rather, energy moves from the zonal mean compartment to and from the eddy compartment.

In the version (xx) of the eddy energy balance, it is traditional to refer to the first term on the RHS as the conversion from zonal to kinetic energy, associated with a local downgradient eddy momentum flux, and the second term as the divergence of an energy flux $\overline{v'p'}$. But this decomposition into conversion and transport is somewhat arbitrary, and we could equally well refer to the first term in (xx) as the conversion and the second term as convergence of an energy flux $\overline{v'p'} + \overline{u}\overline{u'v'}$. (Motivation for referring to this term as an energy flux follows from the fully nonlinear form (), from which one can see that the term $\overline{u}\overline{u'v'}$ emerges from the flux of kinetic energy). One can argue that this latter form is more natural, from the form of the equation for Z, or by considering the case of a steady, unforced, inviscid wave. Recall that for this special case the eddy vorticity flux vanishes but the momentum flux does not, in general. So in such a wave, for which there is no change in time of the eddy kinetic energy, there is a cancellation between non-zero conversion and non-zero energy flux convergence using (), while the two terms on the RHS side of () vanish individually.

We can combine the energy equation for small disturbances with pseudomomentum conservation to obtain

$$\frac{\partial}{\partial t} \int (\mathcal{E} - \overline{u}\mathcal{P}) dy \tag{2.33}$$

The integral of P is conserved, so if some P moves to a region of smaller $\overline{u}$, the integral of $\mathcal{E}$ must decrease. The intergrandin () is referred to as the density of *pseudoenergy*, since a conservation equation of this form can be obtained for analogous problems whenever the basic state on which the waves propagate is independent of time and satisfies the equations of motion.

The reader can check that this discussion of energetics carries over to the sphere. In particular, the spherical version of (xx) is

$$\int_{-\pi/2}^{\pi/2} \mathcal{E} \cos(\theta) d\theta = - \int_{-\pi/2}^{\pi/2} \overline{u'v'} \cos(\theta) \frac{1}{a} \frac{\partial}{\partial \theta} \left(\frac{\overline{u}}{\cos(\theta)}\right) \cos(\theta) d\theta \tag{2.34}$$

On the sphere, eddy energy grows if the angular momentum flux is down the gradient of the angular velocity.

2.5 Food for thought

The fact that the eddy flux of momentum in parts of the troposphere is directed up the angular velocity gradient, so that kinetic energy is transfered from eddies to the mean flow, has resulted in the teminology negative viscosity being applied to these eddies. This term should be avoided in discussions of the atmospheric climate. We have seen that energy transfer from eddy to mean flow occurs whenever a disturbance propagates from stronger westerlies to weaker westerlies, ie., whenever the eddy generation region is localized within the zone of strong westerlies. If the dominant eddy source were localized in the tropics, the eddies would instead extract energy from the mean flow as they propagated into midlatitudes from the tropics.

The term negative viscosity is sometimes used as a synonym for the inverse energy cascade in two-dimensional flow. This inverse cascade will be discussed in lecture ??. But one has to keep in mind that transferring energy to larger scales is not equivalent to transfering energy to the zonal mean flow. Think of the instability of a zonal jet in our two-dimensional homoegenous incompressible flow. One can prove quite generally that as an unforced flow evolves in two dimensions, the mean scale at which the energy resides (as defined by the horizontal wavenumber weighted by the kinetic energy spectrum) must increase (see, for example, Salmon, p. xx). Yet the unstable waves must still extract kinetic energy from the mean flow, this being the only source of energy in this problem. It is useful to try to picture how the energy spectrum of the eddy and zonal mean flow evolve to accomplish this.

It is also interesting to generalize the derivation leading to the relationship between the zonal mean flux of vorticity and the meridional flux of pseudomomentum . Before taking the zonal mean, the poleward vorticity flux is related to the latitudinal derivative of uv and the longitudinal derivative of (v^2u^2). We have seen that uv is related to the meridional flux of pseudomomentum. v^2u^2 is related to the zonal flux of psedomomentum in precisely the same sense. To see this, note that the zonal group velocity of our Rossby wave is

$$G_x = U + \beta(k^2 - \ell^2)/(k^2 + \ell^2)^2 \tag{2.35}$$

The rest of the derivation proceeds as before leading to the generalization:

$$\boldsymbol{v}\zeta \approx \boldsymbol{\nabla} \cdot (\boldsymbol{G}P) \tag{2.36}$$

A stirred barotropic model with linear damping can be used to simulate the observed eddy momentum fluxes in the upper troposphere rather well, as recently discussed by DelSole (2001). Given that linear damping is presumably not a good model for the actual eddy dissipation mechanisms in the upper troposphere, it is surprising how accurate a fit to the observations can be generated in this simple way.

This mixing argument makes the claim that the zonally averaged distribution of surface winds (and the associated surface pressures and mean meridional circulation) is determined by the location of the region in which the system is "stirred" by baroclinic instability. Suppose that the baroclinic excitation happened to be concentrated in two distinct latitude spans per hemisphere, rather than in one, as on Earth. Would there be two separate regions of surface westerlies and a five-cell circulation? The answer is definitely yes. Just such a pattern can emerge in atmospheric models when the eddy size is sufficiently small compared to the radius of the planet, as occurs when the rotation rate is increased. Numerical models clearly show that, given enough room, baroclinic excitation prefers to be organized into distinct, latitudinally confined storm tracks, even when (actually, especially when) the forcing creating the instability varies gradually in latitude. These numerical simulations confirm the correspondence between the preferred regions of excitation, the surface wind distribution, and the mean meridional circulation. The complication is that the spatial pattern of the stirring can be the result of self-organization that does not follow in a self-evident way from the structure of the external forcing.