If you have not already done so, you are strongly encouraged to read the companion file on the non-divergent barotropic vorticity equation, before proceeding to this shallow water case. We do not repeat the discussion that can be found there concerning spherical coordinates and the spherical harmonic spectral transform method.

The Shallow Water Equations

The *shallow water equations* describe the evolution of a hydrostatic homogeneous (constant density), incompressible flow on the surface of the sphere.

The hydrostatic equation is accurate when the aspect ratio of the flow, the ratio of the vertical scale to the horizontal scale, is small. The shallow water equations are only relevant when the horizontal scale of the flow is much smaller than the depth of the fluid.

Hydrostatic balance is the statement that gravity balances the pressure gradient in the vertical equation of motion, implying that vertical accelerations are negligible:

$$\frac{\partial p}{\partial z} = -\rho g \tag{1}$$

If the density ρ is constant, then this equation implies that the horizontal pressure gradient is independent of z. One can therefore look for solutions in which the horizontal flow itself is independent of height. This is the key simplification that underlies the shallow water system. Since the horizontal flow is independent of height, incompressibility implies that the vertical velocity is linear in z.

The shallow water equations are utilized in different contexts in meteorology, with different upper and lower boundary conditions. In the most familiar case of a free upper surface, with imposed constant pressure p_s at the upper surface, integrating the hydrostatic equation down from the top we find that the pressure at height z within the fluid is $p_s + \rho g(h - z)$, where h is the height of the interface. Therefore,

$$\nabla p = \rho \nabla \Phi \tag{2}$$

where $\Phi \equiv gh$. If the lower boundary is flat, then $h = H$, where H is the thickness of the fluid layer. More generally, $H = h - h_M$, where h_M is the height of the lower boundary. In the default version of the code $h_M \equiv 0$.

In the *one and one-half layer* configuration, one assumes that there are two layers of fluid, with densities ρ_1 and ρ_2, with $\rho_2 > \rho_1$. The top of the upper layer is assumed to be at a fixed height, z_T. One assumes that the horizontal pressure gradient in the lower, denser, layer vanishes (as a shorthand, one sometimes just states that the lower layer is "at rest" – it's geostrophic flow is zero at least). Integrating up from a pressure p_2 at some fixed height z_2 in the lower layer, with the interface at $z = \eta$, then at some height z_1 in the upper layer, $p(z_1) = p_2 - (\eta - z_2)\rho_2 g - (z_1 - \eta)\rho_1 g$. The horizontal pressure gradient in the upper layer is then

$$\frac{1}{\rho_1}\nabla p_1 = -g\frac{\rho_2 - \rho_1}{\rho_1}\nabla\eta \equiv -g^*\nabla\eta \tag{3}$$

where g^* is referred to as the *reduced gravity*. We use the notation $\Phi = g^*(z_T - \eta)g^*H$ where H is the thickness once again.

In a rotating frame on the sphere, the horizontal equations of motion are

$$\frac{\partial u}{\partial t} = fv - \frac{u}{a\cos(\theta)}\frac{\partial u}{\partial\lambda} - \frac{v}{a}\frac{\partial u}{\partial\theta} + \frac{uv\tan(\theta)}{a} - \frac{1}{a\cos(\theta)}\frac{\partial\Phi}{\partial\lambda} \tag{4}$$

and

$$\frac{\partial v}{\partial t} = -fu - \frac{u}{a\cos(\theta)}\frac{\partial v}{\partial\lambda} - \frac{v}{a}\frac{\partial v}{\partial\theta} - \frac{u^2\tan(\theta)}{a} - \frac{1}{a}\frac{\partial\Phi}{\partial\theta} \tag{5}$$

A constant radius a appears where the radial coordinate r would appear in full generality, consistent with the assumption of a thin layer of fluid. Vertical advection does not appear because u and v are assumed to be independent of height. The Coriolis and metric terms involving vertical motion are not included so as to maintain energy and angular momentum conservation laws. In a more convenient form,

$$\frac{\partial u}{\partial t} = (f + \zeta)v - \frac{1}{a\cos(\theta)}\frac{\partial(E + \Phi)}{\partial\lambda} \tag{6}$$

and

$$\frac{\partial v}{\partial t} = -(f + \zeta)u - \frac{1}{a}\frac{\partial(E + \Phi)}{\partial\theta} \tag{7}$$

where

$$\zeta \equiv \frac{1}{a\cos(\theta)}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos(\theta)}\frac{\partial}{\partial\theta}\left(u\cos(\theta)\right) \tag{8}$$

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and
$$E \equiv \frac{u^2 + v^2}{2} \tag{9}$$

The vorticity equation is then

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot (\mathbf{v}(f + \zeta)) \tag{10}$$

while the divergence equation is

$$\frac{\partial \mathcal{D}}{\partial t} = -\nabla \cdot \mathbf{A} - \nabla^2(E + \Phi) \tag{11}$$

where

$$\mathcal{D} \equiv \frac{1}{a \cos(\theta)} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos(\theta)} \frac{\partial}{\partial \theta} \left(v \cos(\theta) \right) \tag{12}$$

and

$$\mathbf{A} \equiv (f + \zeta)\mathbf{k} \times \mathbf{v} = (f + \zeta)(v, -u) \tag{13}$$

Throughout these notes, boldface is used for the component of a vector along the surface of the sphere – the radial component, if relevant, is always written separately.

The vorticity equation can be rewritten as

$$\frac{\partial(f + \zeta)}{\partial t} = -\mathbf{v} \cdot \nabla(f + \zeta) - (f + \zeta)\nabla \cdot \mathbf{v} \tag{14}$$

or

$$\frac{D(f + \zeta)}{Dt} = -(f + \zeta)\mathcal{D} \tag{15}$$

where D/Dt is the material derivative.

On the sphere, the vorticity and divergence completely define the flow. To obtain (u, v) from ζ and $\mathcal{D}$, one can first solve the Poisson equations for the streamfunction and velocity potential: $\nabla^2 \psi = \zeta$ and $\nabla^2 \phi = \mathcal{D}$. One then has

$$u = -\frac{1}{a} \frac{\partial \psi}{\partial \theta} + \frac{1}{a \cos(\theta)} \frac{\partial \phi}{\partial \lambda} \tag{16}$$

$$v = \frac{1}{a \cos(\theta)} \frac{\partial \psi}{\partial \lambda} + \frac{1}{a} \frac{\partial \phi}{\partial \theta} \tag{17}$$

To complete the set of shallow water equations the momentum (or vorticity-divergence) equations must be supplemented with a statement of conservation of mass, or a *thickness* equation.

$$\frac{\partial H}{\partial t} = -\nabla\cdot(\mathbf{v}H) \tag{18}$$

or

$$\frac{DH}{Dt} = -H\nabla \cdot \mathbf{v} = -H\mathcal{D} \tag{19}$$

Combining this thickness equation with the vorticity equation, one obtains the famous result that *potential vorticity* is conserved following the flow:

$$\frac{DQ}{Dt} = 0 \tag{20}$$

where

$$Q \equiv \frac{f + \zeta}{H} \tag{21}$$

We have $\Phi \propto H$ whether we are thinking of this shallow water model in the conventional way, with a free surface at constant pressure and a rigid lower boundary, or as the upper layer of a two-layer system in which there is no pressure gradient in the lower layer and the top of the upper layer is rigid. So we can also write

$$\frac{D\Phi}{Dt} = -\Phi\nabla \cdot \mathbf{v} = -\Phi\mathcal{D} \tag{22}$$

and, if we like, we can redefine $Q \equiv (f + \zeta)/\Phi$. Working in terms of Φ, the gravity or reduced gravity does not enter the (unforced) problem. In the code, we use Φ as the prognostic variable, and the potential vorticity is defined in the alternative way as well when it is output for diagnostic purposes.

In summary, a complete set of equations for the unforced and inviscid shallow water equations consists of the equation for u and v, or ζ and $\mathcal{D}$, and Φ.

1 Time differencing

The code uses a standard *semi-implicit* leapfrog scheme, followed by a Robert-Asselin time filter.

The semi-implicit aspect of the algorithm allows one to take time steps that are determined by the strength of the advection, rather that the gravity wave speed, $\sqrt{\Phi}$. This technique is invariably employed in spectral models and is, in fact, in large part responsible for the predominant use of the spectral transform algorithm in global atmospheric modeling. Let T represent the state of the system, and divide the operator N describing the evolution of this system into two parts, $N = N_E + N_I$:

$$\frac{\partial T}{\partial t} = N_E(T) + N_I(T) \tag{23}$$

Use a centered leapfrog scheme for N_E and an implicit scheme for N_I:

$$\frac{T^{i+1} - T^{i-1}}{2\Delta t} = N_E(T^i) + N_I(\alpha T^{i+1} + (1-\alpha)T^{i-1}) \tag{24}$$

We assume that N_I is a linear operator in T, otherwise an expensive iteration would be required to solve this implicit system. The choice of $\alpha = 0.5$, centered implicit, is the one typically used, but one can also consider a fully backward scheme ($\alpha = 1.0$), for example, if one prefers to damp as well as slow down the gravity waves.

Using the notation

$$\xi \equiv 2\alpha\Delta t, \tag{25}$$

and

$$\delta T \equiv \frac{T^{i+1} - T^{i-1}}{2\Delta t} \tag{26}$$

we find

$$\delta T = N_E T^i + N_I(T^{i-1}) + \xi N_I(\delta T) \tag{27}$$

or

$$\delta T = (1 - \xi N_I)^{-1} G \tag{28}$$

where

$$G = N_E T^i + N_I(T^{i-1}) = N(T^i) + N_I(T^{i-1} - T^i) \tag{29}$$

If we linearize the equations about a state of rest with uniform $\Phi = \Phi_0$, and ignore rotation (since we will be treating the Coriolis force explicitly), then the only terms remaining are

$$\frac{\partial \mathcal{D}}{\partial t} = -\nabla^2 \Phi \tag{30}$$

and

$$\frac{\partial \Phi}{\partial t} = -\Phi_0 \mathcal{D} \tag{31}$$

These two equations can be combined into the standard wave equation for the shallow water gravity wave. Treating only these terms implicitly, in the full equations, we can write them in the form,

$$\delta \mathcal{D} = G_{\mathcal{D}} - \xi \nabla^2 \delta \Phi \tag{32}$$

and

$$\delta \Phi = G_{\Phi} - \xi \Phi_0 \delta \mathcal{D} \tag{33}$$

where

$$G_{\mathcal{D}} = N_{\mathcal{D}} - \xi \nabla^2 (\Phi^{i-1} - \Phi^i) \tag{34}$$

$$G_{\Phi} = N_{\Phi} - \xi \Phi_0 (\mathcal{D}^{i-1} - \mathcal{D}^i) \tag{35}$$

Here $N_{\mathcal{D}}$ and N_{Φ} refer to all of the terms of the right hand sides of these equations evaluated at the centered time i. (See below for the algorithm when other sources and sinks are present.)

In a spherical harmonic basis, the Laplacian can be replaced by the appropriate eigenvalue $-\sigma$, $(\sigma > 0)$, so that, for each harmonic we can solve these two simultaneous equations for the divergence tendency and then evaluate the geopotential tendency. Suppressing the spherical modal indices, for each harmonic of the divergence and geopotential tendency we have

$$\delta \mathcal{D} = \frac{G_{\mathcal{D}} + \sigma \xi G_{\Phi}}{1 + \sigma \xi^2 \Phi_0} \tag{36}$$

followed by

$$\delta \Phi = G_{\Phi} - \xi \Phi_0 \delta \mathcal{D} \tag{37}$$

2 Algorithm

Transforms_mod provides several high-level routines for moving from vorticity and divergence to (u, v) and back, as well as for computing the advection operator $-\mathbf{v} \cdot \nabla \xi$. *subroutine uv_grid_from_vor_div* takes as input vorticity and divergence in the spectral domain and returns (u, v) on the grid. It first computes $(cos(\theta)u, cos(\theta)v)$ in the spectral domain, then transforms these to the grid and divides by $cos(\theta)$. *subroutine vor_div_from_uv_grid* reverses this

procedure. *subroutine horizontal_advection* computes $-\mathbf{v} \cdot \nabla \xi$ in the grid domain, given ξ in the spectral domain and (u, v) on the grid. It does this by first computing $cos(\theta) \nabla \xi$ in the spectral domain, transforms the components to grid space, and then divides by $cos(\theta)$ and multiplies by the components of the velocity field in the grid domain. If one also has available the divergence, $\mathcal{D}$, on the grid, one can then compute $-\nabla \cdot (\mathbf{v} \xi) = -\mathbf{v} \cdot \nabla \xi - \xi \mathcal{D}$. One could also compute this quantity using *vor_div_from_uv_grid*. The answer would be identical.

An outline of the steps involved in integrating the shallow water spectral model is as follows:

Assume that we know the spectral and grid vorticity, divergence, and geopotential, and the grid values of u and v, at $t - \Delta t$ and t. Then

1: On the grid, compute tendencies of u, v, and Φ, $(\delta u, \delta v, \delta \Phi)$, due to sources and sinks (computed in the module *shallow_physics_mod*).

2: Compute $(f + \zeta)v$ and $-(f + \zeta)u$ on the grid at time t, and add to δu and δv respectively.

3: compute the divergence and curl of $(\delta u, \delta v)$ to obtain $\delta \zeta$ and $\delta \mathcal{D}$ in the spectral domain, using *subroutine vor_div_from_uv_grid*.

4: add to the grid tendency $\delta \Phi$ the term $-\nabla \cdot (\mathbf{v} \xi)$ as described above, using *subroutine horizontal_advection*, and then convert to spectral domain

5: compute the kinetic energy, E, and add to Φ at t on the grid, transform to spectral domain, take the Laplacian, and add $-\nabla^2(E + \Phi)$ to the spectral divergence tendency $\delta \mathcal{D}$;

6: correct the spectral divergence and geopotential tendencies to take into account the semi-implicit algorithm, as described above

7: add the (nth-)harmonic damping to this advective tendency in the spectral domain, treating the damping implicitly once again, as in the barotropic model

$$\delta \xi \rightarrow \frac{\delta \xi - \nu(2\Delta t)\sigma^n \tilde{\zeta}}{1 + \nu(2\Delta t)\sigma^n} \qquad (38)$$

for $\xi = \zeta, \mathcal{D}, \Phi$, where σ is once again the (absolute value of the) eigenvalue of the Laplacian for each spherical harmonic.

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8: generate the new spectral vorticity, divergence, and geopotential $\xi(t +$ $\Delta t)$ and apply the Robert filter to modify the spectral values at time t.

9: compute grid relative vorticity, divergence, and geopotential – and grid u and v at $t + \Delta t$ from the spectral vorticity and divergence using *subroutine uv_grid_from_vor_div*

Note that we do not bother to compute the new grid u, v, Φ at time t resulting from the Robert filter. The extra transforms do not seem to be needed to maintain smooth temporal evolution in these shallow water simulations, since the memory resides in the spectral domain.

There is an option of carrying a passive scalar tracer, using explicit spectral advection and harmonic dissipation. For development purposes, there is also an option of carrying along another passive tracer which is a grid-point variable advected using a piecewise linear finite volume technique.

The finite volume advection scheme follows closely that described in

Lin, S.-J. and R. B. Rood, 1996: Multidimensional flux-form semi-Lagrangian transport schemes. Monthly Weather Review, 124, pp. 2046-2069.

and is discussed further in the documentation for the module *fv_advection_mod*. This scheme is not particularly suited for use in a leapfrog context. The procedure used here carries two time levels of information for the tracer ξ, just as for vorticity. Using the velocities at time t, we call the finite volume advection algorithm to advect the scalar ξ from $t - \Delta t$ to $t + \Delta t$; then we Robert filter ξ. There is no explicit diffusion in this scheme.

The integrations are started with a simple forward step of length Δt.

3   Default example

By default the programs starts from a state of rest, in the rotating frame, with uniform Φ, and a flow is generated by relaxing Φ to $\Phi_{eq}(\lambda, \theta)$. The velocities are also relaxed linearly to zero. We set

$$\Phi_{eq} = \Phi_0 + Q_{Monsoon}e^{-(x^2+y^2)} + Q_{ITCZ}e^{-d^2} \tag{39}$$

where

$$x = \frac{\lambda - \lambda_M}{2W_M}; \;\; y = \frac{\theta - \theta_M}{W_M}; \;\; d = \frac{\theta}{W_I} \tag{40}$$

Isolating the ITCZ mass source results in a zonally symmetric Hadley cell response. Isolating the monsoonal source results in a subtropical anticyclone that can be unstable if forced strongly enough. If both the ITCZ and monsoonal sources are present, as in the default settings, one also generates a quasi-stationary Rossby wave emanating from the monsoonal source.

See the barotropic model description for a discussion of appropriate choices for spectral and grid resolution. The default is a standard T85 model.

4   Structure of the code

There is a generic main program, in *main.f90*, that is used in a number of idealized atmospheric models. Besides doing some bookkeeping, it includes the main time loop and has a namelist in which the time step Δt and the length of the integration are provided.

The main program runs the model by calling the routines in *atmosphere_mod*, which, in turn, use routines in *shallow_dynamics_mod*, *shallow_physics_mod*, *shallows_diagnostics_mod*, and *fv_advection_mod*. *shallow_physics* contains a namelist with which one controls the forcing in the default example. Model resolution, the spectral damping, and the strength of the time filter are controlled by a namelist read by *shallow_dynamics_mod*.

The dynamics module uses *transforms_mod*, which contains a variety of routines for transforming data from spherical harmonics to a grid and back, computing derivatives in the spectral domain, etc.

Diagnostics are controlled by the diagnostics manager. Those immediately available are $u, v, \Phi, \zeta, \mathcal{D}, f + \zeta, \psi, Q, \xi_s, \xi_g$ where the latter two are the gridded tracer fields generated, respectively, using spectral advection and finite-volume advection. All fields are output into netcdf files, one per processor, that can automatically be combined into full spatial fields using FMS's *mppnccombine* utility. The time interval at which output is generated, file names, etc are controlled from the *diag_table* file read by the diagnostics manager. To add additional diagnostic fields, follow the template in *shallow_diagnostics_mod*.