1. Transport and mixing

1.1 The material derivative

Let $\boldsymbol{V}(\boldsymbol{x}, t)$ be the velocity of a fluid at the point $\boldsymbol{x} = (x, y, z)$ and time t. Consider also some scalar field $\chi(\boldsymbol{x}, t)$ such as the temperature or density. We are interested not only in the partial derivative of χ with respect to time, $\partial\chi/\partial t$, but also in the time derivative following the motion of the fluid, $d\chi/dt$. The latter is the so-called *material derivative*:

$$\begin{aligned} d\chi/dt &= \lim_{\delta t \to 0}[\chi(\boldsymbol{x} + \boldsymbol{V}(\boldsymbol{x}, t)\delta t, t + \delta t) - \chi(\boldsymbol{x}, t)]/\delta t \\ &= (\partial/\partial t + \boldsymbol{V} \cdot \nabla)\chi \end{aligned} . \tag{1.1}$$

In Cartesian coordinates, $\boldsymbol{V} = (u, v, w)$, $\nabla = (\partial_x, \partial_y, \partial_z)$ and

$$d\chi/dt = \partial\chi/\partial t + u\partial\chi/\partial x + v\partial\chi/\partial y + w\partial\chi/\partial z . \tag{1.2}$$

We will also be using spherical coordinates. The notation (u, v, w) is conventional for the (eastward, northward, radially outward) components of the velocity field. In addition to the radial distance from the origin, r, we use the symbol θ for latitude ($-\pi/2$ at the south pole, $\pi/2$ at the north pole) and λ for longitude (ranging for zero to 2π). The gradient operator in these coordinates is

$$\nabla = \hat{\lambda}(r\cos\theta)^{-1}(\partial/\partial\lambda) + \hat{\theta}r^{-1}(\partial/\partial\theta) + \hat{\boldsymbol{r}}(\partial/\partial r), \tag{1.3}$$

so that

$$d/dt = \partial/\partial t + (r\cos\theta)^{-1}u(\partial/\partial\lambda) + r^{-1}v(\partial/\partial\theta) + w(\partial/\partial r) . \tag{1.4}$$

The material derivative of a vector, such as the velocity $\boldsymbol{V}$ itself, is defined just as in 1.1. But care is required when considering the material derivative of a component of a vector if the unit vectors of one's coordinate system are position dependent, as they are in spherical coordinates. For example, the radial component of the material derivative of the velocity is not equal to the material derivative of the radial component of the velocity; rather,

$$\hat{\boldsymbol{r}} \cdot \frac{d\boldsymbol{V}}{dt} = \frac{d}{dt}(\hat{\boldsymbol{r}} \cdot \boldsymbol{V}) - \boldsymbol{V} \cdot \frac{d\hat{\boldsymbol{r}}}{dt} . \tag{1.5}$$

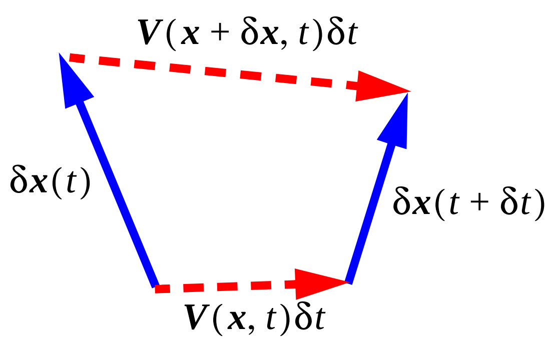
The second term on the rhs is sometimes referred to as the *metric term*. After obtaining expressions for $d\hat{\boldsymbol{r}}/dt$, etc., one arrives at the spherical-coordinate expression for the acceleration of a fluid parcel (see Problem 1.1):

$$\frac{d\boldsymbol{V}}{dt} = \left\{ \begin{array}{l} \hat{\lambda}\left(\dfrac{du}{dt} - uv\tan\theta/r + uw/r\right) \\ + \hat{\theta}\left(\dfrac{dv}{dt} + u^2\tan\theta/r + vw/r\right). \\ + \hat{\boldsymbol{r}}\left(\dfrac{dw}{dt} - (u^2+v^2)/r\right). \end{array} \right. \tag{1.6}$$

Another case of interest is the material derivative of an infinitesimal material line segment. Suppose that $\boldsymbol{x}$ and $\boldsymbol{x} + \delta\boldsymbol{x}$ are infinitesimally close to each other, with the vector $\delta\boldsymbol{x}$ pointing from one point to the other, and let $\delta\boldsymbol{x}(t)$ be the evolution of this vector assuming that its end-points move with the flow. Then, from the figure below, $d\delta\boldsymbol{x}/dt = (\delta\boldsymbol{x}\cdot\nabla)\boldsymbol{V}$. To make sure that we understand this notation, suppose that the line segment is oriented vertically at the time in question. Then, in Cartesian coordinates,

$$d\delta x/dt = \delta z(\partial u/\partial z); \quad d\delta y/dt = \delta z(\partial v/\partial z); \quad d\delta z/dt = \delta z(\partial w/\partial z).$$

We refer to the third term as stretching, as it lengthens or shortens the line segment, and the other two terms as tilting, as they change the orientation of the line segment without changing its length. This equation and the terminology will be important when we discuss vorticity dynamics.



The fractional rate of the change of the length of a infinitesimal vector pointing in the x-direction, moving with the fluid, is $\partial u/\partial x$ and similarly for y and z. Therefore, the fractional rate of change in the volume of an infinitesimal piece of fluid per unit time is

$$\begin{aligned} (1/\delta V)d\delta V/dt &= (1/\delta x\delta y\delta z)d(\delta x\delta y\delta z)/dt \\ &= (1/\delta x)d\delta x/dt + (1/\delta y)d\delta y/dt + (1/\delta z)d\delta z/dt \\ &= \partial u/\partial x + \partial v/\partial y + \partial w/\partial z \end{aligned} \tag{1.7}$$

in Cartesian coordinates, or, more generally,

$$(1/\delta V)d\delta V/dt = \nabla\cdot\boldsymbol{V}. \tag{1.8}$$

With this physical interpretation in mind, the expression for the divergence in spherical coordinates can be seen to take the form

$$\nabla \cdot \boldsymbol{V} = (r\cos\theta)^{-1}\frac{\partial u}{\partial \lambda} + (r\cos\theta)^{-1}\frac{\partial[(\cos\theta)v]}{\partial\theta} + r^{-2}\frac{\partial(r^2 w)}{\partial r}, \tag{1.9}$$

using $\delta V \propto r^2 \cos\theta dr d\theta d\lambda$.

1.2 Conservation of mass

If $\rho(\boldsymbol{x}, t)$ is the density of the fluid, the statement that mass is neither created nor destroyed can be written in two equivalent forms:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{V}) \tag{1.10}$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \boldsymbol{V}. \tag{1.11}$$

The first (flux) form is more convenient when considering a fixed volume in space, since, using Gauss's Theorem, it immediately yields

$$\frac{\partial}{\partial t}\iiint_V \rho dV = -\iint_S \rho \boldsymbol{V} \cdot \hat{\boldsymbol{n}} dA, \tag{1.12}$$

where V is any volume, S its surface, and $\hat{\boldsymbol{n}}$ a unit vector normal to S and directed outwards. The latter (advective) form is more convenient when following the fluid, and expresses the fact that the divergence $\nabla \cdot \boldsymbol{V}$ is equal to the fractional change, per unit time, in the density of an infinitesimal parcel of fluid: $d(\rho\delta V)/dt = 0 \quad \Rightarrow \quad (1/\rho)d\rho/dt + (1/\delta V)d\delta V/dt = 0$.

For any other scalar field χ with source S per unit mass, we can write the equation in either advective form,

$$\frac{d\chi}{dt} = S, \tag{1.13}$$

or flux form,

$$\frac{\partial(\rho\chi)}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} \chi) = \rho S. \tag{1.14}$$

1.3 Eulerian vs. Lagrangian descriptions of fluid flow

Consider a point moving with the velocity field $\boldsymbol{V}$. We refer to such a point as a fluid particle. If $\boldsymbol{a}$ is the position of this particle at time t_0, its position at any later time is

$$\boldsymbol{x}(\boldsymbol{a}, t) = \int_{t_0}^{t} \boldsymbol{V}(\boldsymbol{x}(\boldsymbol{a}, t'), t')dt'. \tag{1.15}$$

Rather than consider fields such as $\boldsymbol{V}$ and χ as functions of $\boldsymbol{x}$ and t (the *Eulerian* description), one can instead consider these fields to be functions of $\boldsymbol{a}$ and t: $\chi(\boldsymbol{x}(\boldsymbol{a}, t), t) = \chi_L(\boldsymbol{a}, t)$. The material derivative $d\chi/dt$ is then simply $\partial\chi_L/\partial t$. This *Lagrangian* description is occasionally convenient because of its closer analogy with the classical mechanics of point masses. While we will be working with the Eulerian equations of motion exclusively, it will often be important, if not critical, to maintain a Lagrangian *perspective*, to think not only about fields evolving in $\boldsymbol{x}$ and t, but also about the trajectories of fluid particles and how quantities of interest evolve following the flow.

Simple Eulerian flows do not necessarily produce simple particle trajectories. Given the velocity field $\boldsymbol{V}$, the advection of a passive tracer is formally a linear problem: $\partial\chi/\partial t = -\boldsymbol{V} \cdot \nabla\chi$. Yet this linear problem can also be solved by first solving the nonlinear problem of computing trajectories: $d\boldsymbol{x}/dt = \boldsymbol{V}(\boldsymbol{x}, t)$. The latter problem is identical in form to that of a general dynamical system, except that here the "phase space" of the system is three-dimensional physical space. Autonomous systems (steady flow, in this context) with three degrees of freedom can produce chaotic, extremely complex, trajectories. Non-autonomous systems (unsteady flows) in two dimensions can possess a similar level of complexity. In contrast, autonomous systems in two dimensions cannot produce trajectories more complicated than limit cycles, that is, simple closed streamlines.

Consider a 2-dimensional flow with Cartesian components (u, v) that is non-divergent in the *x*-*y* plane. Such a flow is efficiently described in terms of a streamfunction, ψ, where $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$. Note also that this is a Hamiltonian system with ψ as the Hamiltonian and *x* and *y* as conjugate variables. We analyze 2-dimensional flows more extensively in Chapter 4.

1.4 Diffusion and friction

In the atmosphere there is a tremendous disparity between the scales of motion that contain the bulk of the energy in the flow, on which our interest will be focused, and scales at which molecular diffusion and viscosity become significant.

If we consider some field χ that is advected by the flow $\boldsymbol{V}$ as well as diffused with the constant kinematic diffusivity D, then

$$\partial\chi/\partial t = -\boldsymbol{V} \cdot \nabla\chi + D\nabla^2\chi . \tag{1.16}$$

If the characteristic scale of the velocities in the flow is U, then for variations in χ on the scale L, the characteristic ratio of diffusion to advection is measured by the *Peclet number,*

$$\frac{|\boldsymbol{V} \cdot \nabla\chi|}{|D\nabla^2\chi|} \sim \frac{UL}{D} \equiv \mathrm{Pe} . \tag{1.17}$$

The time required to diffuse away a feature of scale L in the absence of flow is L^2/D; Pe is the ratio of this diffusive time scale to the advective time L/U. If χ is one component of the velocity field itself, so that D is replaced by the kinematic viscosity ν, then the corresponding ratio is UL/ν, the *Reynolds number.*

The order of magnitude of ν or D in the atmosphere near the earth's surface is $10^{-5}\ \mathrm{m^2/s}$. Large-scale flows in the earth's atmosphere typically have velocity scales of $10\ \mathrm{m/s}$ and vertical scales larger than $1\ \mathrm{km}$, leading to Peclet or Reynolds numbers at least of the order of 10^9. A weak gust of wind near the surface with a speed of $1\ \mathrm{m/s}$ and a modest vertical scale of $10\ \mathrm{m}$ still has advection dominating diffusion or viscosity by a factor of 10^6. Only on scales of a millimeter or so do the latter come into play directly. Molecular diffusivities in the ocean are even smaller (see Problem 1.4).

One should not assume on this basis that molecular diffusion and dissipation can be ignored. The atmosphere is a forced, dissipative system through which energy is flowing. The generation of kinetic energy integrated over the depth of the atmosphere is estimated to be 2 - $3\ \mathrm{W/m^2}$, averaged over the globe. Dissipation by molecular viscosity must balance this generation in a steady state. The kinetic energy per unit mass of the atmosphere is roughly $10^2\ \mathrm{m^2/s^2}$ on average, and the total mass of an atmospheric column is $10^4\ \mathrm{kg/m^2}$, so one might estimate that this energy, if not constantly replenished, would be dissipated (converted into heat) within a few days.

In fact, this naive estimate of spin-down time for the atmosphere is misleading, since much of this kinetic energy is intimately related to a part of the potential energy of the flow, and would be replenished by this potential energy as it decayed. Since this potential energy reservoir is an order of magnitude larger than the kinetic energy, a better estimate of a spin-down time for the atmospheric circulation is an order of magnitude longer.

Yet this does not imply that the energy level in the atmosphere is dependent on the precise value of the molecular viscosity of air. On the contrary, the hope and expectation is that a theory for the circulation of the atmosphere would not involve the value of the molecular viscosity in any significant way. This expectation is based on the analogy with fully developed three-dimensional turbulence at very high Reynolds numbers, for which it is found that the rate at which energy *cascades* to small scales determines the rate of dissipation, while the value of the viscosity simply determines the scale at which the dissipation takes place. In fact, this is more than an analogy, since all significant dissipation in the atmosphere (below heights of $\approx 100\ \mathrm{km}$ at least), is presumed to occur in patches of fully developed turbulence.

1.5 Eddy fluxes and turbulent diffusion

In describing and analyzing a complex fluid flow we define various kinds of averages: time averages at a fixed point; averages over one or more spatial dimensions at a fixed time; or coarse-grain averages obtained by smoothing over a certain horizontal scale. For any variable χ we shall write the average in question as $[\chi]$ and the deviations from this average, sometimes referred to as *eddies*, as $\chi' = \chi - [\chi]$.

Consider time-averaging for example. Suppose that χ is a scalar conserved following the flow, except for the source/sink S, and assume that the flow is non-divergent, $\nabla \cdot \boldsymbol{V} = 0$, for simplicity. In a statistically steady state, averaging over time and neglecting molecular diffusion, we have

$$0 = -[\boldsymbol{V}] \cdot \nabla[\chi] - \nabla \cdot [\boldsymbol{V}'\chi'] + [S]\,. \qquad (1.18)$$

There will be a three-way balance among the source/sink S, the advection by the mean flow $[\boldsymbol{V}]$, and the convergence of the transient eddy flux $[\boldsymbol{V}'\chi']$.

If the transient eddy flux were negligible, one could hope to understand the structure of $[\chi]$ from knowledge of $[S]$ and the mean flow $[\boldsymbol{V}]$. Steady-state models can often illuminate aspects of the atmospheric circulation, and are rather common in oceanography. While this is convenient at times, it is rarely the case that the transient eddies are negligible, however, and they often dominate the entire problem.

As another illustration of averaging, consider averages over longitude, an important construction in discussions of the general circulation. If the earth's surface were uniform, there would be no physical distinction between one longitude and another, and the earth's climate would be independent of longitude, that is, zonally symmetric. In reality there are substantial zonal asymmetries in the climate, due to the land-ocean configuration and the topography of the land surface. But in a rough first approximation, one can still think of the climate as varying primarily with latitude, rather than longitude, and one can hope to interpret this latitudinal structure using theoretical models in which the climate is zonally symmetric. It is with these ideas in mind that we are often moved to decompose variables of interest into their zonal means and the departure from the zonal mean.

Consider an incompressible flow once again, and ignore spherical geometry for simplicity. The advection equation for a tracer with source/sink S, averaged over x, reduces to

$$\begin{aligned} \partial[\chi]/\partial t &= -[v]\partial[\chi]/\partial y - [w]\partial[\chi]/\partial z - \partial[v'\chi']/\partial y - \partial[w'\chi']/\partial z + [S] \\ &= -[\boldsymbol{V}] \cdot \nabla[\chi] - \nabla \cdot [\boldsymbol{V}'\chi'] + [S] \end{aligned}\,. \qquad (1.19)$$

In the second line the vectors are two-dimensional, in the y-z plane.

In the case of a nearly conservative tracer, for which the source/sink S is negligible in the region of interest, one is often tempted to think of the eddy fluxes as providing a downgradient turbulent *eddy* diffusion, at least if the flow is sufficiently complex -- i.e.,

$$[\boldsymbol{V}'\chi'] \approx -D\nabla[\chi]\,. \qquad (1.20)$$

The terminology is based on an analogy with molecular diffusion, with eddies of fluid replacing molecules as the transporting agents. The dimensions of kinematic diffusivity D are length^2/time, and one estimates the effective diffusivity with the product of a characteristic velocity V and a mixing length L, or a velocity squared and the inverse of a characteristic time scale $\tau = L/V$.

One can make this idea a bit more precise in a simple special case. Suppose that initially the flow and the χ field are purely zonally symmetric, and that the flow is perturbed by zonal asymmetries (u', v', w') for $t > 0$. Focus on the particle that resides at the point $\boldsymbol{x}$ at time t, and trace it back to its location at $t = 0$. Let this point be $\boldsymbol{x} - \xi(\boldsymbol{x}, t)$, so that $\xi(\boldsymbol{x}, t)$ is the displacement of this particle in time *t*. If ξ is small compared to the spatial scale over which the initial $[\chi]$ field varies, we can set

$$\chi' = [\chi](\boldsymbol{x} - \xi) - [\chi](\boldsymbol{x}) \approx -\xi_j \partial[\chi]/\partial x_j . \tag{1.21}$$

(We use the standard summation convention, with repeated indices automatically summed over, unless otherwise stated.) Therefore,

$$[V_i'\chi'] = -D_{ij}\partial[\chi]/\partial x_j , \tag{1.22}$$

where $D_{ij} = [V_i'\xi_j]$. If the off-diagonal elements of $\mathbf{D}$ happen to be zero, and if the diagonal elements are equal ($D_{ij} = d\delta_{ij}$), then the eddy flux has the form of a simple diffusion: $[V_i'\chi'] = -d\partial[\chi]/\partial x_i$. If the diagonal elements are unequal ($D_{ij} = d_i\delta_{ij}$), the "diffusion" coefficient d_i will be different in the *y* and *z* directions. (Since the atmosphere and the ocean are both confined to a spherical shell that is very thin compared to its radius, atmospheric and oceanic flows are often profoundly anisotropic.)

Since $\boldsymbol{V}' = d\xi/dt$, the diffusivity d_i can be written in the form

$$d_i = [(d\xi_i/dt)\xi_i] = [(d/dt)(\xi_i^2/2)] \qquad \text{(no sum over } i\text{)} . \tag{1.23}$$

Therefore, d_i is positive if fluid particles at the latitude in question are in the process of dispersing (in the *i*'th direction) from their original "home" latitudes; it is negative if the particles are, on average, returning home.

In "turbulent" flows one expects particles to disperse systematically, producing downgradient transport. Such flows are strongly dissipative, and one cannot expect χ to be conserved indefinitely. Characteristically, the fluid particle loses its "memory" in occasional intense mixing events, in which the turbulent cascade brings molecular diffusion into play. If the typical distance traveled between these events in some direction is Δ, then the diffusivity will have the magnitude $V\Delta$, where V is the rms velocity in that direction.

In general, the tensor $\mathbf{D}$ also has off-diagonal elements. In this case, one can still split $\mathbf{D}$ into symmetric and antisymmetric parts, $\mathbf{D} = \mathbf{D}^s + \mathbf{D}^a$:

$$D_{ij}^s = (D_{ij} + D_{ji})/2 ; \qquad D_{ij}^a = (D_{ij} - D_{ji})/2 . \tag{1.24}$$

Once can then rotate in the *y*-*z* plane so as to diagonalize $\mathbf{D}^s$. The part of the transport due to $\mathbf{D}^s$ can then be described as "diffusive", with the eigenvalues of the rotated tensor being the diffusivities. The antisymmetric part ($D_{yz} = -D_{zy} = \Psi$) leads to the flux convergence

$$\begin{aligned} \nabla \cdot [\boldsymbol{V}'\xi'] &= \frac{\partial}{\partial y}(\Psi\partial[\chi]/\partial z) - \frac{\partial}{\partial z}(\Psi\partial[\chi]/\partial y) \\ &= -\boldsymbol{W} \cdot \nabla[\chi] , \end{aligned} \tag{1.25}$$

where $\boldsymbol{W} = (-\partial\Psi/\partial y, \partial\Psi/\partial z)$. This part of the flux is decidedly not diffusive in character; in fact, Ψ acts as the streamfunction in the *y-z* plane for the flow $\boldsymbol{W}$ that advects, rather than diffuses, the mean tracer field. One can then think of $[\chi]$ as being advected by the total flow $[\boldsymbol{V}] + \boldsymbol{W}$. The non-diffusive eddy transport $\boldsymbol{W}$ plays an important role in discussions of the general circulation of the atmosphere and in theories for the incorporation of "mesoscale" eddy fluxes in ocean models.

Mixing and transport play central roles in theories of climate. The atmosphere transports heat down the temperature gradient from the tropics to the poles. In the extratropics (poleward of $30°$ latitude), atmospheric heat transport has something of a diffusive character, the dominant eddies being the cyclones and anticyclones familiar from weather maps. These eddies have a characteristic diffusivity scale of $\approx 10^7\ \mathrm{m^2/s}$. To first approximation, the competition between the resulting poleward heat flux and the north-south gradient in the radiative heating controls the equator-to-pole temperature gradient on the earth.

The interplay between advection and (turbulent) diffusion is at the heart of many problems in meteorology and oceanography, and can be subtle (see problem 1.5).

Problems

1.1 Derive expressions for the material derivatives of the unit vectors in spherical coordinates, and use this result to verify the expression (1.6) for the acceleration in spherical coordinates. You may want to practice on 2-d polar coordinates.

1.2 Consider the following flow

$$u = \Lambda z$$
$$v = V \sin[k(x - ct)]$$

with Λ a positive constant. This is schematic of the flow in the mid-latitude troposphere -- a westerly flow ($u > 0$) that increases with height ($\Lambda > 0$), superposed on a transverse wave (with wavenumber $k \approx 1/1000\ \mathrm{km}^{-1}$) moving eastward ($c > 0$), but less rapidly than the zonal wind at the tropopause, $z = z_T$ ($c < \Lambda z_T$). Consider particles that are located along the $y = 0$ axis at $t = 0$. Compute the position of this line of particles at some later time *t*. Compare with the streamfunction for this flow at the same time. The contrasting behavior in the upper and lower troposphere is important for the dynamics of midlatitude storms.

1.3 As an important illustration of the distinction between Eulerian and Lagrangian perspectives (in a flow for which the particle trajectories retain some simplicity), let $\boldsymbol{V} = (u, v, 0)$ in Cartesian coordinates, where

$$u = A(y)\sin(\omega t), \qquad v = A(y)\cos(\omega t).$$

The time mean of this flow at a fixed point in space is identically zero. In the special case that A is independent of *y*, all fluid particles move clockwise in circular orbits of radius A/ω. For an arbitrary function $A(y)$, find an approximate expression for the time-averaged drift of a particle,

$$\lim_{t \to \infty} \boldsymbol{x}(\boldsymbol{a}, t)/t.$$

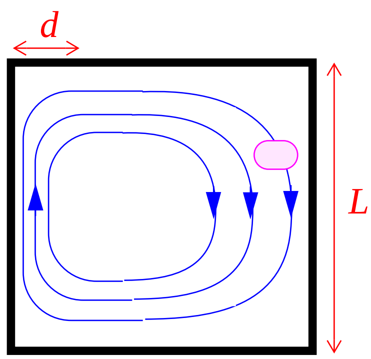
Assume that variations in A are suitably “small”. Be precise as to how small dA/dy must be for your expression to be accurate.

1.4 Look up the value of molecular diffusion of heat in water and estimate how long it would take for a temperature perturbation to mix from the bottom of the ocean to the top, due only to molecular diffusion.

1.5 (Box diffusion) Consider the 2D, non-divergent flow pictured below in a square of size *L*. The flow has a boundary-layer character, meant to resemble the vertically integrated horizontal ocean circulation. The equatorward flow is spread more or less uniformly over the domain, but the poleward flow is concentrated in a layer of width $d \ll L$. The typical size of the velocity in the interior of the domain is *V*. Now consider the advection-diffusion equation:

$$\frac{\partial T}{\partial t} = -\boldsymbol{V} \cdot \nabla T + K\nabla^2 T.$$

Suppose that the initial condition is a spot of tracer as shown in the figure. Estimate how long it will take to homogenize the tracer distribution, as a function of VL/K.



1.6 Solve the advection-diffusion equation exactly for the flow $(\Lambda y, 0, 0)$ and the initial condition $T = A\cos(kx)$. Provide a physical explanation for the long-time behavior of the solution.

Getting started with Problem 1.6:

$$\frac{\partial T}{\partial t} = -\Lambda y \frac{\partial T}{\partial x} + D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right).$$

The *x*-dependence in initial state is separable:

$$T = \mathrm{Re}\{A(y, t)e^{ikx}\}.$$

Equation for *A* then looks like

$$\frac{\partial A}{\partial t} = -C(y)A + D\frac{\partial^2 A}{\partial y^2}.$$

Integrating factor is exp($C(y)t$). Carry on.