6. Barotropic dynamics

Barotropic models do not allow the creation of new vorticity or the conversion of potential energy, but they capture other important phenomena, like external waves and nonlinear interaction, very well. They also possess a shear instability that is analogous to baroclinic shear instability. Barotropic models are relatively simple to analyze and interpret. They were used exclusively in the earliest days of numerical weather prediction, and global barotropic weather forecasts were still used operationally as late as the 1980s.

6.1 Geostrophic adjustment

The high-frequency motion filtered from QG and other balanced models can be understood as "geostrophic adjustment". Consider the linearized shallow-water equations on an f-plane:

$$\frac{\partial \boldsymbol{V}}{\partial t} = -f_0 \hat{\boldsymbol{z}} \times \boldsymbol{V} - g \nabla h \tag{6.1}$$

$$\frac{\partial h}{\partial t} = -H \nabla \cdot \boldsymbol{V}. \tag{6.2}$$

Assume disturbances of the form

$$u = U \exp[i(kx + ly - \nu t)], \tag{6.3}$$

etc. For the existence of non-trivial solutions, we find that

$$\nu(\nu^2 - f_0{}^2 - gHK^2) = 0, \tag{6.4}$$

where $K^2 \equiv k^2 + l^2$. The solution corresponding to $\nu = 0$ is in geostrophic balance. The other solutions have $(\nu/f_0)^2 = 1 + L_R^2 K^2$, where $L_R \equiv \sqrt{gH}/f_0$, the Rossby radius of deformation. The high-frequency solutions are called "gravity-inertia waves" or (in SW model only) "Poincare′ waves". In the atmosphere, these are associated with the so-called "mesoscale", where geostrophic adjustment takes place (confusingly, in the ocean, "mesoscale" refers to a part of the *balanced* motion).

Write $L = 2\pi/K$ for wavelength. If $L \gg L_R$, the motion takes the form of inertial oscillations, $\nu \approx \pm f_0$:

$$\partial \boldsymbol{V}/\partial t = -f_0 \hat{\boldsymbol{z}} \times \boldsymbol{V}, \tag{6.5}$$

with h constant. These do not disperse energy and are therefore not effective in adjusting the flow towards balance. If $L \ll L_R$, the adjustment is in form of gravitational oscillations, $\nu \approx \pm K\sqrt{gH}$:

$$\partial \boldsymbol{V}/\partial t = -g \nabla h, \tag{6.6}$$

with $\partial h/\partial t = -H \nabla \cdot \boldsymbol{V}$. Note that the condition $\mathrm{Ro} \ll 1$, with Ro defined as in chapter 5, does not imply geostrophic balance, since the time scale of linear waves is $\tau = L/c$, not L/V, which is much larger. In the case of inertial waves, small Ro is actually the criterion for linearity .

After the high-frequency motion disperses, we expect, from chapter 5, that most of the adjustment will have occurred in the flow field (as opposed to the mass field) if the length scales are larger than the Rossby radius; at small scales, it is mainly the mass field that adjusts to the flow field. With $H = 10^4\ \mathrm{m}$, one finds that $L_R \approx 3500\ \mathrm{km}$ in the atmosphere. In the ocean, the “internal” Rossby radius (with gravity effectively reduced by the relative size of density variations) is at most 100 km .

6.2 Barotropic energetics

Multiply 6.1 by $\boldsymbol{V}$ to get an equation for *kinetic energy:*

$$\frac{\partial}{\partial t}\left(\frac{|\boldsymbol{V}|^2}{2}\right) = -\boldsymbol{V}\cdot\nabla(gh)\,; \tag{6.7}$$

and 6.2 by *gh/H* to get an equation for *potential energy:*

$$\frac{\partial}{\partial t}\left(\frac{gh^2}{2H}\right) = -(gh)\nabla\cdot\boldsymbol{V}\,. \tag{6.8}$$

The terms on the rhs convert kinetic energy to potential energy or vice-versa. One can see this by adding the two equations:

$$\frac{\partial E}{\partial t} = -\nabla\cdot(\boldsymbol{V}gh)\,, \tag{6.9}$$

where $E \equiv |\boldsymbol{V}|^2/2 + gh^2/(2H)$, the total energy. The rhs of 6.9 is a flux convergence that integrates to boundary radiation. If a region has no net radiation at its boundaries, the area integral of E over that region is conserved and the rhs of 6.7 and 6.8 can only convert between kinetic and potential. (The latter is more accurately called “available potential energy”; the energy associated with the mean depth H is not available for conversion).

Alternative derivation *for the QG model*: start with the potential vorticity equation for the quasi-geostrophic SW model:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}_g\cdot\nabla\right)\left(\nabla^2 h - \frac{f_0^{\,2}}{gH}h\right) = 0\,. \tag{6.10}$$

Multiply by h and manipulate to reach:

$$\frac{\partial}{\partial t}\left(\frac{|\nabla h|^2}{2} + \frac{f_0^2}{gH}\frac{h^2}{2}\right) = -\nabla\cdot\left[\boldsymbol{V}_g\left(\frac{|\nabla h|^2}{2} + \frac{f_0^2}{gH}\frac{h^2}{2}\right) - h\frac{d_g}{dt}\nabla h\right], \tag{6.11}$$

where d_g/dt is the operator in 6.10. Recalling $\boldsymbol{V}_g = (g/f_0)\hat{\boldsymbol{z}}\times\nabla h$, we see that the energy -- in parentheses on the lhs of 6.11 -- is the same as in 6.9 except for a constant factor and the assumption of geostrophic velocity. The second part of the flux on the rhs can be interpreted using the QG momentum equation.

In the linearized model, it is easy to show that, for a given PV distribution, the state with the least total energy is the one in geostrophic balance. This implies that geostrophic adjustment always *removes* energy (see problem 6.2). From 6.11, we see that the ratio of potential to kinetic energy in a geostrophically balanced flow is $E_P/E_K \approx L^2/L_R^2$. This is consistent with previous remarks about the "burden of change" during geostrophic adjustment.

6.3 Non-divergent barotropic model

If the flow of a homogeneous fluid is purely non-divergent in the horizontal, then the vertical motion vanishes and the vorticity equation reduces to

$$\frac{d\zeta}{dt} = -\beta v . \tag{6.12}$$

We refer to this as the "barotropic vorticity equation". In the meteorological context, its use is sometimes justified by considering "equivalent barotropic" flows, as follows.

Equivalent barotropic flows have the structure $\varphi = A(z)\langle\varphi\rangle$, where the angle brackets indicate vertical averaging and A is the vertical profile. On large scales, the atmosphere often looks like this. The QG vorticity equation is

$$\frac{d_g \zeta_g}{dt} = -\beta v_g + f_0 \frac{\partial w}{\partial z} . \tag{6.13}$$

If we assume equivalent barotropic structure and integrate between levels where w vanishes, we get

$$\frac{\partial\langle\zeta_g\rangle}{\partial t} + \langle \boldsymbol{V}_g\rangle \cdot \nabla(\langle A^2\rangle\langle\zeta_g\rangle + \beta y) = 0 , \tag{6.14}$$

where $\langle\zeta_g\rangle = \hat{\boldsymbol{z}} \cdot \nabla\times\langle\boldsymbol{V}_g\rangle$ and we have used the fact that $\langle A\rangle = 1$. The constant factor $\langle A^2\rangle$ appears only in the quadratic term. Thus, multiplying 6.14 by this factor and defining $\zeta^* \equiv \langle A^2\rangle\langle\zeta\rangle$, etc., leads to

$$\frac{\partial\zeta_g^*}{\partial t} + \boldsymbol{V}_g^* \cdot \nabla(\zeta_g^* + \beta y) = 0 . \tag{6.15}$$

This is the same as 6.12. The variables denoted by asterisks may be interpreted as the original variables evaluated at the level z^* where $A = \langle A^2\rangle$. This is called the "equivalent barotropic level". Typically, z^* is near mid-troposphere. The result 6.15 depends on the very strong assumption that $A(z)$ exists (with no horizontal variation).

The PV for this barotropic model is simply $\zeta_g^* + f$. If non-divergence is stipulated *a priori*, the QG approximation is not needed, since the total velocity is $\boldsymbol{V} = \nabla_\perp\psi$ (non-divergence takes the place of a mass-momentum balance relation). Then the PV is $\zeta + f$. The total energy reduces to the kinetic energy: $E \equiv |\boldsymbol{V}_g|^2/2$ or $E \equiv |\boldsymbol{V}|^2/2$.

6.4 Rossby Waves

As seen in chapter 5, the quasi-geostrophic SW model becomes non-divergent in the limit $L \ll L_R$, where L_R is the Rossby radius. The vorticity equation is then 6.12. What is the effect of β on small-amplitude perturbations to a uniform zonal flow in this model?

To get the phase speed of such disturbances, assume a plane-wave perturbation to the zonal flow:

$$\psi = -Uy + A\sin k(x - ct). \tag{6.16}$$

Then $\zeta = -Ak^2 \sin k(x - ct)$. Substituting into 6.12 leads to the dispersion relation:

$$c = U - \frac{\beta}{k^2}. \tag{6.17}$$

The waves are known as barotropic “Rossby waves”. Propagation is *westward* relative to *U*.

Stationary Rossby waves are possible: c is zero when the wavelength $L = 2\pi/k$ has the value

$$L_s = 2\pi\sqrt{\frac{U}{\beta}}. \tag{6.18}$$

With $\beta \approx 1.6\times10^{-11}\ \mathrm{m^{-1}s^{-1}}$ and $U = 25\ \mathrm{ms^{-1}}$, expect $L_s \approx 8000\ \mathrm{km}$ in the atmosphere. The intrinsic Rossby-wave phase speed becomes comparable to the gravity-wave phase speed when $k^2 = \beta/\sqrt{gH}$. For $\sqrt{gH} = 100\ \mathrm{ms^{-1}}$, the corresponding wavelength L is about 15,000 km.

6.5 Topographic β-effect

The vorticity equation for the shallow-water model can be written

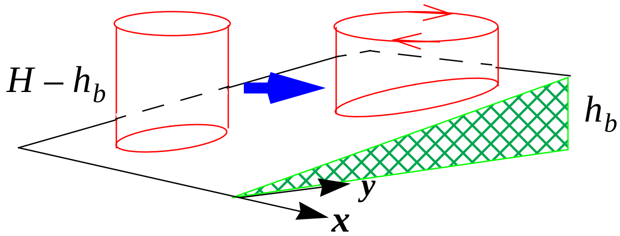
$$h\left(\frac{d\zeta}{dt} + \beta v\right) = (\zeta + f)\frac{dh}{dt}, \tag{6.19}$$

where $h(x, y)$ is the *total* depth of the fluid. Suppose that changes in h are partly due to bottom topography, *i.e.*, $h = H + h' - h_b$, where h_b is height of the topography and H is the mean depth. We can make the topographic effect resemble the β-effect by linearizing 6.19 and assuming the bottom height depends only on *y*. Thus, with $h_b \ll H$ and $|\zeta| \ll |f| \approx f_0$,

$$\frac{d\zeta}{dt} = -\left(\beta + \frac{f_0}{H}\frac{dh_b}{dy}\right)v + f_0 w/H \tag{6.20}$$

(since $dh/dt = -v\partial h_b/\partial y$). The bottom slope simply modifies β.

The linearization of bottom topography is implicit in the quasi-geostrophic model, because the constraint $h_b \ll H$ is necessary to maintain order-Ro divergence.



6.6 Divergent barotropic model

When $f = f(y)$, small-amplitude solutions of the SW equations are a mixture of gravity-inertia waves and Rossby waves. In the case of no basic flow, the linearized SW system is

$$u_t = fv - gh_x \tag{6.21}$$

$$v_t = -fu - gh_y \tag{6.22}$$

$$h_t = -H(u_x + v_y). \tag{6.23}$$

Assume solutions of the form $u = U(y)\exp[ik(x-ct)]$, $v = V(y)\exp[ik(x-ct)]$, etc. Eliminating h in 6.21-6.23 yields

$$-ikc(1 - gH/c^2)U = fV - (gH/c)V'. \tag{6.24}$$

An obvious solution is simply $V \equiv 0$, with $c = \pm\sqrt{gH}$. From 6.21-6.22 one then deduces that

$$U(y) = U_0 \exp\left(-\int_{0_0}^{y} \frac{f}{c} dy'\right). \tag{6.25}$$

Note that $|c/f| = L_R(y)$, the local Rossby radius (for waves near the equator, the appropriate scale, L_E, is defined below). These solutions, known as Kelvin waves, require walls to support the exponential structure. This in turn constrains the sign of the phase speed. In the NH, the waves propagate with the wall on the right (clockwise around an island).

For all other solutions, we eliminate u to reach

$$\frac{d^2V}{dy^2} + \left(\frac{c^2k^2 - f^2}{gH} - k^2 - \frac{\beta}{c}\right)V = 0. \tag{6.26}$$

Make the mid-latitude β-plane approximation by replacing f^2 with f_0^2 and taking β constant. Then $V = V_0 \exp(ily)$ and 6.26 becomes a cubic equation for c in terms of k and l. The first two eigenvalues are

$$c_{GI} \approx \pm\sqrt{gH(1 + l^2/k^2) + f_0^2/k^2}. \tag{6.27}$$

These “gravity-inertia” wave phase speeds are the same as the high-frequency solutions of 6.4, since $c = \nu/k$. The third eigenvalue is

$$c_R \approx -\beta/[k^2 + l^2 + f_0^2/(gH)]\,, \tag{6.28}$$

which is the same as 6.17, except for an extra term in the denominator due to horizontal divergence. The approximations in 6.27 and 6.28 are based on the assumption that $\sqrt{gH} \gg \beta/k^2$. That is, the Rossby modes are well separated from the gravity modes.

Now make the *equatorial* β-plane approximation by replacing f^2 with $\beta^2 y^2$ in 6.26. Then solutions with $V \neq 0$ take the form

$$V = H_n(\xi)\exp(-\xi^2/2)\,, \tag{6.29}$$

where $\xi = y/L_E$, with $L_E \equiv \sqrt{(gH)^{1/2}/\beta}$, and H_n is the Hermite polynomial of the n-th order: $H_0 = 1$, $H_1 = 2\xi$, $H_2 = 4\xi^2 - 2$, $\ldots$. The frequency equation is

$$\frac{c^2k^2}{gH} - k^2 - \frac{\beta}{c} = \frac{2n+1}{L_E^2}, \qquad n = 0, 1, 2, \ldots. \tag{6.30}$$

The first two solutions are

$$c_{GE} \approx \pm\sqrt{gH\left(1 + \frac{2n+1}{k^2L_E^2}\right)}, \tag{6.31}$$

corresponding to eastward and westward propagating equatorial gravity waves. The third is

$$c_{RE} \approx -\beta/\left(k^2 + \frac{2n+1}{L_E^2}\right), \tag{6.32}$$

corresponding to (westward-propagating) equatorial Rossby waves. In comparison to 6.27 and 6.28, the meridional wavenumber has been replaced by $\sqrt{2n+1}/L_E$. The meridional variation 6.29 can be characterized as a damped oscillation in y with more or less structure depending on n.

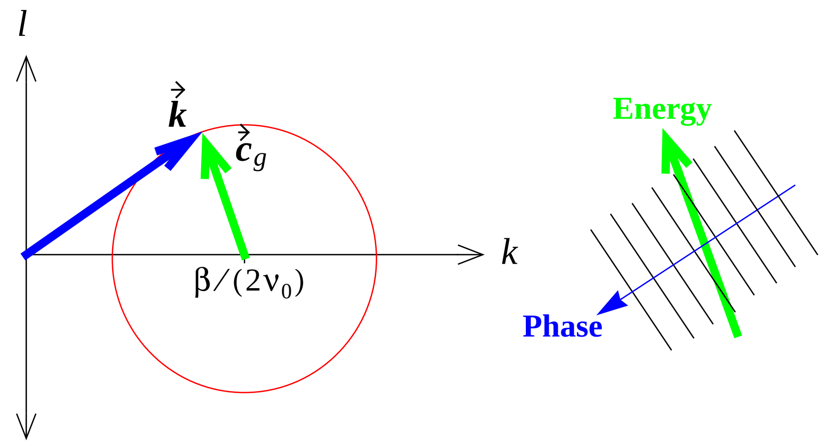
Equatorial Kelvin waves have $V = 0$ and $U = U_0\exp(-\frac{1}{2}y^2/L_E^2)$, consistent with 6.25. They propagate only one way, towards the east, with $c = \sqrt{gH}$.

6.7 Dispersion and reflection of Rossby waves

The dispersion relation 6.28 for Rossby waves in the quasi-geostrophic SW model can be written:

$$\nu = \frac{-\beta k}{k^2 + l^2 + L_R^{-2}}. \tag{6.33}$$

For real wavenumbers k and l, the frequency lies in the range $-\beta L_R/2 < \nu < 0$. For a given $\nu = -\nu_0$ in this range, wavenumbers satisfying 6.33 lie on a circle in the (k, l) plane centered at $k = \beta/(2\nu_0)$ and $l = 0$.



Wave energy moves according to $E_t = -\boldsymbol{c}_g \cdot \nabla E$, where $\boldsymbol{c}_g(k, l)$ is the group velocity:

$$\boldsymbol{c}_g = \frac{\partial \nu}{\partial k}\hat{\boldsymbol{x}} + \frac{\partial \nu}{\partial l}\hat{\boldsymbol{y}}. \tag{6.34}$$

Think of the above graph as though the coordinates were physical distance, so that $\boldsymbol{k} = k\hat{\boldsymbol{x}} + l\hat{\boldsymbol{y}}$, parallel to the direction of phase propagation in physical space (diagram at right). Then the group velocity may be written

$$\boldsymbol{c}_g = \frac{2\nu_0^2}{\beta k}\left(\boldsymbol{k} - \frac{\beta}{2\nu_0}\hat{\boldsymbol{x}}\right). \tag{6.35}$$

The vector in parentheses is indicated in the left-hand diagram (labelled $\boldsymbol{c}_g$). Notice that it has the same amplitude for all wavenumbers. The rectified energy $\langle E \rangle$ is $(K^2 + L_R^{-2})(A^2/4)$, where A is the amplitude of the streamfunction. Hence, in view of 6.33 and 6.35,

$$\boldsymbol{c}_g\langle E \rangle = \frac{\nu_0}{2}A^2\left(\boldsymbol{k} - \frac{\beta}{2\nu_0}\hat{\boldsymbol{x}}\right), \tag{6.36}$$

showing that the amplitude of the energy flux depends only on the amplitude of the streamfunction and the frequency.

Reflections of Rossby waves at a western wall preserve both ν and l. This constraint together with the direction of energy propagation determines the zonal wavenumber for the incident and reflected waves. From the diagram, one can see that the incident wave, with westward group velocity, has the shorter $\boldsymbol{k}$ and larger scale. At a southern or northern wall, k is preserved and the scales are the same. For the energy, the angle of incidence always equals the angle of reflection.

6.8 Pseudomomentum

Instead of β, we now use a meridionally varying *relative* vorticity in the basic state. Consider a purely zonal flow $u = U(y)$. A small perturbation (denoted by primes) satisfies

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u' = -v'\frac{dU}{dy} + fv' - \frac{\partial p'}{\partial x}, \quad (6.37)$$

so that the perturbation energy E is governed by

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)E = -u'v'\frac{dU}{dy} + (\text{flux convergence}). \quad (6.38)$$

The conversion term on the rhs is due to eddy momentum flux ("Reynolds stress") in the presence of mean shear. The perturbation energy grows when the flux is "downgradient" (towards smaller values of U). The flux direction is recognizable in the tilt of the eddy phase lines, since

$$u'v' = -\psi_x\psi_y = \left.\frac{dy}{dx}\right|_\psi u'^2. \quad (6.39)$$

By defining the meridional displacement η such that $d\eta/dt = v'$, we notice that the momentum flux may also be written

$$u'v' = -\eta\frac{dU}{dy}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta. \quad (6.40)$$

Then from 6.38 and 6.40,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[E - \frac{\eta^2}{2}\left(\frac{dU}{dy}\right)^2\right] = (\text{flux convergence}). \quad (6.41)$$

As the eddy energy grows in the presence of shear, so does the mean squared displacement of the fluid particles.

The eddy vorticity equation is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta' = -v'\frac{d\bar{\zeta}}{dy}, \quad (6.42)$$

whence the eddy "enstrophy" equation is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\zeta'^2}{2} = -v'\zeta'\frac{d\bar{\zeta}}{dy}. \quad (6.43)$$

The nonlinear zonal momentum equation may be written

$$u_t = fv - (uu)_x - (uv)_y - p_x. \quad (6.44)$$

The zonal average of this is $\bar{u}_t = -(\overline{u'v'})_y$. Upon zonally averaging 6.43 as well, we find that

$$\frac{\partial}{\partial t}\left(\bar{u} + \frac{\overline{\zeta'^2}}{2d\bar{\zeta}/dy}\right) = 0. \quad (6.45)$$

The second term in parentheses is a quadratic measure of the disturbance, equivalent to $\frac{1}{2}\eta^2 d\bar{\zeta}/dy$. Essentially because of 6.45, it is known as the *pseudomomentum*. The result shows concisely that wherever the eddy field is growing (for whatever reason), maxima and minima of $\bar{u}$ are flattened out over time ($\bar{u}$ has a maximum/minimum where $d\bar{\zeta}/dy$ is positive/negative). Eq. 6.45 also follows from Stokes' Theorem, as applied to the regions on each side of a latitude circle from which particles are displaced meridionally by a wave.

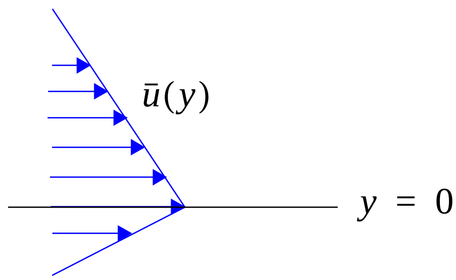
6.9 Barotropic shear waves

Normal modes are easy to obtain for piecewise-linear basic wind profiles. The vorticity is constant in each of a finite number of layers. At the interfaces, the vorticity and/or wind may be discontinuous. Smooth wind profiles may be usefully approximated by piecewise linear ones as long as one is not specifically interested in “critical layers”, where $\bar{u} = c$.

The simplest case is

$$\bar{u} = u_0 - \begin{cases} \bar{\zeta}_2 y, & y \geq 0 \\ \bar{\zeta}_1 y, & y \leq 0 \end{cases}, \tag{6.46}$$

with $\bar{\zeta}_1$ and $\bar{\zeta}_2$ both constant. As seen below, the wind is continuous at $y = 0$. Assume a distur-



bance of the form $\psi = \tilde{\psi}(y)\exp[ik(x-ct)]$. Then the barotropic vorticity equation 6.12 implies

$$(\bar{u}-c)\left(-k^2 + \frac{d^2}{dy^2}\right)\tilde{\psi} = -\tilde{\psi}\frac{d\bar{\zeta}}{dy}. \tag{6.47}$$

Off the interface at $y = 0$, the rhs vanishes. Solutions within the two layers are therefore either $\tilde{\zeta} = a\delta(y-y_0)$ with $c = \bar{u}(y_0)$ (“continuous spectrum”) or $\tilde{\psi} = a\exp(\pm ky)$ (“discrete spectrum”). In the second case, we reject the outwardly growing solutions in each region and write

$$\tilde{\psi} = \begin{cases} a_2\exp(-ky), & y \geq 0 \\ a_1\exp(ky), & y \leq 0 \end{cases} \tag{6.48}$$

for our eigenmode.

Two independent matching conditions are required at $y = 0$. The appropriate quantities to match are particle displacement and pressure. The former is $\eta = \psi/(\bar{u}-c)$, which requires continuity of the streamfunction when $\bar{u}$ is continuous. The *x*-momentum equation directly implies

$$p' = (\bar{u}-c)\psi_y + \left(f - \frac{d\bar{u}}{dy}\right)\psi, \tag{6.49}$$

where p' is the pressure perturbation. The second matching condition is, therefore, that the rhs of 6.49 should be continuous. Note that since f and ψ are continuous, the Coriolis force is irrelevant.

Application of the matching conditions to 6.48 yields $a_1 = a_2$ and

$$c = u_0 - \frac{\bar{\zeta}_2 - \bar{\zeta}_1}{2k}, \tag{6.50}$$

since $d\bar{u}/dy = -\bar{\zeta}$. We see that all normal modes are neutral (c is real). Disturbances of this type are known as "edge waves". Compare 6.50 to the Rossby-wave dispersion relation for a one-dimensional disturbance in the absence of shear: $c = u_0 - \beta/k^2$.

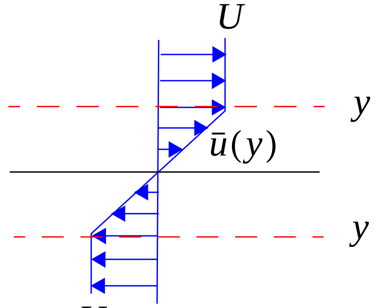
Growing normal modes are possible if another piece is added to the profile. Consider the flow

$$\bar{u} = \begin{cases} U, & y \geq L \\ Uy/L, & -L \leq y \leq L \\ -U, & y \leq -L, \end{cases} \tag{6.51}$$

known as a shear-layer profile (see diagram below). Eigenmodes from the discrete spectrum are of the form

$$\tilde{\psi} = \begin{cases} a_2 \exp(-ky), & y \geq L \\ b_1 \exp(ky) + b_2 \exp(-ky), & -L \leq y \leq L \\ a_1 \exp(ky), & y \leq -L. \end{cases} \tag{6.52}$$

Since 6.51 is continuous, we again require ψ to match across each of the two interfaces, $y = \pm L$.



Continuity of pressure in the same places then leads to

$$\xi^2 = \kappa^{-2}[(1-\kappa)^2 - \exp(-2\kappa)], \tag{6.53}$$

with $\xi = c/U$ and $\kappa \equiv 2kL$. For $\kappa < 1.28$ (the short-wave "cutoff"), we find that $\xi^2 < 0$, implying growing or damping solutions and no propagation relative to the flow at $y = 0$. Shorter waves than this are neutral, with $\xi^2 \to 1$ in the limit $\kappa \to \infty$. The long-wave limit, $\kappa \to 0$, corresponds to a basic state with a single vortex "sheet" at $y = 0$. Rayleigh (1894) solved this case directly to find that $\xi^2 = -1$, which can be confirmed from 6.53 using a Taylor-series expansion of the exponential.

For short waves, $(a_2/a_1)^2 \approx (1+\xi)/(1-\xi)$, where the rhs is real and positive. Hence there is no tilt to the neutral waves. Their amplitude becomes concentrated at whichever interface has $\bar{u} \approx c$. However, for the long waves, we find $(a_2/a_1)^2 \approx -1$, implying a meridional tilt. The tilt of the growing modes is against the shear, as required by 6.38.

6.10 Necessary conditions for instability

The existence or non-existence of growing disturbances in a given zonal flow can be anticipated from general considerations. Start with the 6.47, multiply by $\tilde{\psi}^*/(\bar{u}-c)$ and integrate over y to reach

$$\int\left(-k^2|\tilde{\psi}|^2+\tilde{\psi}^*\frac{d^2}{dy^2}\tilde{\psi}\right)dy = -\int\frac{|\tilde{\psi}|^2}{\bar{u}-c}\left(\frac{d\bar{\zeta}}{dy}\right)dy. \qquad (6.54)$$

Upon integrating by parts on the lhs and assuming $d\tilde{\psi}/dy \rightarrow 0$ at infinity, we get

$$\int\left(k^2|\tilde{\psi}|^2+\left|\frac{d\tilde{\psi}}{dy}\right|^2\right)dy = \int\frac{|\tilde{\psi}|^2}{|\bar{u}-c|^2}(\bar{u}-c_r+ic_i)\left(\frac{d\bar{\zeta}}{dy}\right)dy, \qquad (6.55)$$

where we have also written $c = c_r + ic_i$ for the complex phase speed.

•*Rayleigh-Kuo Theorem:*

The imaginary part of this relation is

$$0 = c_i\int\frac{|\tilde{\psi}|^2}{|\bar{u}-c|^2}\frac{d\bar{\zeta}}{dy}dy. \qquad (6.56)$$

The quantity $|\tilde{\psi}|^2/|\bar{u}-c|^2 = |\tilde{\eta}|^2$ is positive definite. It follows that for unstable modes ($c_i \neq 0$) to exist, the vorticity gradient $d\bar{\zeta}/dy$ must have both signs within the flow. This result is known as the Rayleigh-Kuo theorem.

For smooth profiles, a sign change in $d\bar{\zeta}/dy$ implies that the velocity profile has an *inflection point*, that is, $d\bar{\zeta}/dy = -d^2\bar{u}/dy^2 = 0$ somewhere. Accordingly, instabilities in this type of flow are know as “inflectional” instabilities. Notice that the sign change guarantees the existence of at least one pair of “counter-propagating” neutral modes (in the short-wave limit), since the phase speed of neutral waves is proportional to $-d\bar{\zeta}/dy$ (*cf*. 6.50).

•*Fjortoft’s Theorem:*

The real part of 6.55 is

$$\text{lhs} = \int\frac{|\tilde{\psi}|^2}{|\bar{u}-c|^2}\left(\bar{u}\frac{d\bar{\zeta}}{dy}\right)dy, \qquad (6.57)$$

where we have used 6.56 with $c_i \neq 0$. Since the lhs is positive definite, 6.57 implies that the velocity $\bar{u}$ must be positively correlated with the vorticity gradient. This conclusion -- known as Fjortoft’s Theorem -- implies that $\bar{u}$ is negatively correlated with the neutral phase speed associated with the local vorticity gradient. One can understand inflectional instabilities as a phase locking and constructive interference between a pair of counter-propagating edge waves. Fjortoft’s Theorem implies that, for instability, the counter-propagation must be resisted by the mean shear.

Problems

6.1 Derive the energy principle for the nonlinear shallow-water model, Eqs. 5.1-5.3, in

a) Lagrangian form: $dE_1/dt = -\nabla \cdot \boldsymbol{F}_1$; and

b) Eulerian form: $\partial E_2/\partial t = -\nabla \cdot \boldsymbol{F}_2$.

6.2 Linearize the *f*-plane shallow-water system assuming no basic flow. Suppose that a localized initial disturbance is given in terms of its Fourier components as:

$$(\psi, \phi) = \iint (\tilde{\psi}_{kl}, \tilde{\phi}_{kl}) e^{i(kx + ly)} dk dl,$$

where ψ is the streamfunction and ϕ is the geopotential.

a) Obtain the geopotential field at the completion of geostrophic adjustment.

b) Show that the change in total energy during the adjustment is always negative (or zero). Neglect any initial divergence.

Note: it is sufficient to work with a single Fourier component. Characterize all results in terms of the scale $L_R = \sqrt{gH}/f_0$.

6.3 Consider an *axisymmetric* barotropic shear flow specified by

$$\zeta = \begin{cases} 2\Omega, & r \le R \\ 0, & r > R \end{cases}$$

for constants Ω and R. The tangential velocity is continuous. Perturb the boundary of the circular region of constant vorticity with a sinusoidal disturbance. Find the phase speed of the resulting edge wave in terms of its azimuthal wavenumber. Show how this result reduces to the dispersion relation 6.50 as the wavenumber increases.

6.4 Steady solutions of the barotropic vorticity equation linearized about a constant zonal flow $u = U$ and variable bottom topography $h = h_b(x, y)$, in the presence of a linear drag, satisfy

$$U\frac{\partial}{\partial x}\left(\zeta' + \frac{f_0}{H}h_b\right) = -\beta v' - r\zeta',$$

where r is a positive constant.

a) Solve for the “Green’s function” solution corresponding to $h_b = h_0\delta(x)\cos(ly)$. This solution decays away from the topographic source in both the positive and negative *x*-directions. Write the solution in closed form for non-zero but “small” *r*. (Wherever necessary, assume that *r* is sufficiently small to simplify the algebra.) Graph the meridional flow perturbation $v'(x, y = 0)$. Assume that $0 < U < \beta / l^2$ and $\beta > 0$.

b) Relate the damping distance in $x > 0$ to the zonal group velocity c_{gx} and damping time $\tau = 1/r$. Is there a similar interpretation of the upstream damping scale?