1. Oscillations in a sealed tube.

A disk of mass $M$ just fits into an upright tube of height $H_0$ and cross-section $A$ that is open at the top. The ambient air density and pressure are approximately constant, say $\rho_0$ and $p_0$, respectively. The acceleration due to gravity is $g$ and the air is an ideal gas.

a) The disk is placed in the tube and allowed to come to rest. Find an expression for the equilibrium height $H$ of the disk, assuming that the air in the tube is in thermal equilibrium with the surroundings, that the disk forms an air-tight seal with the sides of the tube, and that the effects of friction can be neglected. Hint: start by relating $H$ to the change in density in the sealed part of the tube assuming $H$ is much smaller than the density scale height.

b) Now suppose the tube is perfectly insulated so that the potential temperature $\theta_0$ is preserved? Is the equilibrium level higher or lower, and by how much? Hint: the pressure now varies as $(\rho/\rho_0)^\gamma$ and $\gamma \neq 1$.

c) Assume that the disk is now perturbed slightly from its equilibrium position. What is the frequency of its oscillation in each of the scenarios (a) and (b)
d) Finally, suppose $M = \rho_M h A$ where $\rho_M$ is the density of the disk. Discuss the special case where $\rho_M = \rho_0$ and $h = H$. In particular, how is $h$ related to the properties of a linear sound wave?

2. Flow in a shallowing channel.

Consider a straight, shallow channel with constant width $L$ but bottom elevation rising from $z = 0$ at the far left to $z = h$ at the far right. There is a rigid lid at $z = H > h$. A homogeneous, inviscid fluid flows from left to right, entering with depth $H$ and uniform speed $U$.

a) Determine the profile $u(y)$ at the far right when: i) the channel is not rotating; ii) the channel is on an $f$-plane.

b) In terms of $h/H$, what is the maximum nondimensional width $L(f/U)$ that allows a steady flow without any changes in the upstream conditions?
3. Scale analysis in a rotating atmosphere.

Consider a rapidly rotating Boussinesq fluid on an \( f \)-plane. “Rapidly rotating” means the rotation period is short compared to the advective time scale.

a) Restricting attention to horizontal variations, show that the pressure divided by the density scales like \( \phi \sim fUL \).

b) Split the horizontal velocity into a “geostrophic” part \( V^g = f^{-1} \nabla^\perp \phi \) and an “ageostrophic” residual \( V^a = V - V^g \). Show that the ratio of magnitudes of \( V^a \) to \( V^g \) is estimated by \( U/(fL) \).

c) Show that the scaling \( W \sim UH/L \) is an overestimate for the magnitude of the vertical velocity by deriving the correct estimate. What is the weakest condition on \( H/L \) to ensure hydrostatic balance?

d) For the buoyancy, write \( b = N^2 z + b' \), with \( N^2 \) constant. The corresponding pressure decomposition is \( \phi = N^2 z^2/2 + \phi' \) and you already have a scale for \( \phi' \). Find two scales for \( b' \) -- one based on hydrostatic balance and one based on buoyancy conservation. Use these to show that \( L \sim NH/f \).

END